CGN 6933-002
Transport in Porous Media

Spring 2005 — University of South Florida
Homework #6 — Civil & Environmental Eng.
Due Wed., March 23, 2005 — J. A. Cunningham

(1) Consider the transport of conservative bromide tracer through a very long (semi-infinite) column of porous medium. The transport can be described with the following partial differential equation.

$$\frac{\partial C(x,t)}{\partial t} = D\,\frac{\partial^2 C(x,t)}{\partial x^2} - v\,\frac{\partial C(x,t)}{\partial x}$$

Suppose that the column is initially devoid of any bromide mass, but that beginning at time $t = 0$, the bromide concentration at the front end of the column ($x = 0$) is held constant. Then we can write the following initial and boundary conditions.

$$C(x, t = 0) = 0$$

$$C(x = 0, t \geq 0) = C_0$$

$$C(x \to \infty, t) = 0$$

For these initial and boundary conditions, there is an analytical solution to the advective-dispersive equation. (See, for instance, Fetter, p. 61.)

$$C(x,t) = \frac{C_0}{2}\left(\operatorname{erfc}\left(\frac{x - v\,t}{2\sqrt{D\,t}}\right) + \exp\left(\frac{v\,x}{D}\right)\operatorname{erfc}\left(\frac{x + v\,t}{2\sqrt{D\,t}}\right)\right)$$

(a) Suppose that, for our particular semi-infinite column, we have a porosity $n = 0.30$, a specific discharge $q = 1.5$ cm/d, a dispersion coefficient $D = 8.0$ cm^2/d, and a concentration $C_0 = 100$ mg/L. For these conditions, plot the concentration profiles (i.e., $C(x,t)$ vs. x) for times $t = 1$ d, $t = 3$ d, and $t = 5$ d. Put all three curves on the same graph. Hint: you might want to do it in MatLab, because it might make things easier for some of the following problems.

(b) Repeat the exercise from part (a), but use a dispersion coefficient $D = 0.33$ cm^2/d. In a few sentences, compare the profiles from part (a) to the profiles from part (b).

(2) Suppose that, instead of a semi-infinite column, we have a column that is $L = 50$ cm long. The transport of bromide through this column is still described by the advective-dispersive equation, and suppose that we have the same initial condition (i.e., no bromide in the column at time $t = 0$). However, for our finite column, we have the following boundary conditions.

$$C(x = 0, t \geq 0) = C_0$$

$$C(x = L, t) = 0$$

For these boundary conditions, there is no analytical solution to the advective-dispersive equation. If we want to graph the concentration profiles now, we will have to solve the PDE numerically rather than analytically.

(a) Write a finite-difference program to determine the concentration profiles at times $t = 1$ d, $t = 3$ d, and $t = 5$ d. Use an explicit ("forward Euler") scheme in your program. Use the physical parameters from part (a) of problem (1). Also, use the following parameters for your program: spatial discretization $\Delta x = 1$ cm, time step $\Delta t = 0.05$ d. Plot the three concentration profiles on the same graph.

(b) Compare the profiles from problem (1a) to those from problem (2a). Do you think it is OK to use the analytical solution as an approximate solution for problem (2)? Why or why not?

(c) Increase your time step Δt to 0.1 d. Re-run your program. What happened? Why?

(3) In this problem, you will compare explicit and implicit methods.

(a) Re-do part (a) of problem (2), but this time use an implicit ("backward Euler") scheme. Use a time step $\Delta t = 0.05$ d. How do the profiles look compared to those from (2a)? In other words, can you see a difference between the explicit and implicit methods for $\Delta t = 0.05$ d?

(b) Now increase your time step Δt to 0.1 d. Re-run your program. What happened this time? Compare the implicit and explicit methods for the case that $\Delta t = 0.1$ d.

(4) In this problem, you will use a Crank-Nicolson method, and you will examine the effect of dispersion on numerical solutions.

(a) Re-do part (a) of problem (2), but this time use a Crank-Nicolson scheme. Use a time step $\Delta t = 0.05$ d. How do the profiles look compared to those in problems (2a) and (3a)? Is there an obvious difference between the three methods?

(b) Decrease the dispersion coefficient to $D = 0.33$ cm^2/d. Re-run your program. Plot the results for $t = 1$ d and $t = 5$ d. Compare to the profiles from problem (1b). What happens to the numerical solution? Why? To facilitate your explanation, you might want to calculate the relevant Peclet numbers.

(5) About how long (measured in hours) did it take you to complete this homework?