Multi-Operational Machining Processes Modeling for Sequential Root Cause Identification and Measurement Reduction

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1 Introduction

In a multi-operational machining process, part variation at each operation may be due to an accumulation of operational errors. When the part and process are complicated, variation reduction is often constrained by the measurement capability for providing sufficient data. Examples include engine machining processes where the number of features is extremely large relative to measurement resources. A complete measurement of one engine cylinder head may require 1–2 h of expensive time on Coordinate Measurement Machine (CMM), which is considerably long comparing with process throughput. Therefore, what product features and/or process variables are to be measured, has been studied by a number of researchers [1–5]. The proposed strategies include: (1) optimizing the Fisher information matrix to reduce sensing noise [1–4]; (2) making system response robust to process parameters [1]; and (3) improving process diagnosability [4–6]. However, little research discussed how the model formulation impacts measurement synthesis and how the measurement constraint can be considered in the early stage of modeling.

Modeling variation propagation has been proved to be an effective way for variation reduction and design synthesis in multi-operational manufacturing processes. The available model formulation includes time series model [7,8], state space models [9–14], and state transition model [15]. A brief review is given to the previously developed state space model.

For an N-operation manufacturing process, the state of the kth operation x(k) is described as a linear combination of the previous state x(k−1), process input u(k), and natural process variation z(k). Quality characteristic y(k) is a linear transformation of state x(k) plus measurement noise η(k). Under small deviation assumption, the model has the following form [9–14]

\[ x(k) = A(k-1)x(k-1) + B(k)u(k) + z(k), \quad k = 1, 2, \ldots, N, \]

\[ y(k) = C(k)x(k) + η(k), \quad \{k \in \{1, 2, \ldots, N\}. \] (1)

For machining processes, state vector x(k) represents the deviations of part features. The process deviation u(k) includes fixture and machine tool deviations, while the datum deviation is contained in x(k−1). State transition matrix A(k−1) and input coefficient matrix B(k) are constant matrices determined by product and process design. C(k) is determined by measurement design. Denote by y the quality characteristics of N operations and by u the process deviations from all operations. The relationship between y and u can be obtained by solving Eq. (1), which ends up as a linear model in the form of \( y = Γu + e \). Diagnosis and measurement synthesis can be performed by analyzing the rank \( Γ \) [3,5,16]. The problem encountered, however, is that \( Γ \) is often not full rank for machining processes. One natural thought is to increase the dimension of quality characteristics y to increase the rank of \( Γ \) matrix. Nevertheless, this strategy cannot guarantee the full rank of \( Γ \) because datum, fixture, and machine tool errors could generate the same error patterns on part features. Previously developed approaches for machining processes [11–13], however, did not directly model this process physics. Consequently, it is difficult to distinguish error sources at each operation [17].

The strategy proposed in this paper is to formulate the variation propagation model using the proposed equivalent fixture error (EFE) concept. With this concept, datum error and machine tool error are transformed to equivalent fixture locator errors at each operation. As a result, error sources can be grouped and root cause identification can be conducted in a sequential manner. The case studies demonstrate the model validity through a real cutting experiment and model advantage in measurement reduction for root cause identification. [DOI: 10.1115/1.1948403]
The paper is organized as follows. Section 2 introduces some preliminaries and notations. Detailed derivation of the new variation propagation model is conducted in Sec. 3. Section 3 also presents the concept of EFE and condition of grouping three error sources. The case studies in Sec. 4 demonstrate the validity of equivalent fixture error through a real cutting experiment and model advantage in measurement reduction for root cause identification. Conclusions and future research work are discussed in Sec. 5.

2 Preliminaries and Notations

This section briefly introduces part and machining operation models developed in previous literature [11–13,18].

Using vectorial surface model [19], an $M$-surface part $X(k)$ after operation $k$ is represented as a vector in the part coordinate system (PCS)

$$X(k) = \left( X_1^j(k), \ldots, X_j^j(k), \ldots, X_M^j(k) \right)^T,$$

where $X_j^j(k)$ denotes the $j$th surface and it is represented as

$$X_j^j(k) = (v_j^j(k) p_j^j(k) r_j^j(k))^T = (v_j^j(k) v_j^j(k) p_j^j(k) p_j^j(k) p_j^j(k) r_j^j(k))^T,$$

where $v_j^j(k) = (v_j^j(k) v_j^j(k) v_j^j(k))$ and $p_j^j(k) = (p_j^j(k) p_j^j(k) p_j^j(k))^T$, and $r_j^j(k)$ are orientation, location and size of surface $j$, respectively. Subscripts $x$, $y$, and $z$ denote three directions in the coordinate system. $M$ is determined by product design and process planning. The size of cylindrical hole can be represented by the radius of the hole and size of plane is zero.

The nominal surface $j$ and part are denoted as $X_0^j(j)$ and $X_0^j(j)$, respectively. The deviation of $X_j^j(k)$ is denoted as $x_j^j(k) = X_j^j(k) - X_0^j(j) = (\Delta x_j^j(k) \Delta y_j^j(k) \Delta z_j^j(k))^T$. Accordingly, the part deviation after operation $k$ is denoted as $x_k^j(k) = (x_1^j(k), \ldots, x_j^j(k), \ldots, x_M^j(k))^T$.

Operations involved in machining mainly include setup and cutting. Since the part is modeled as a vector, operations and their errors can be viewed as vector transformations. Therefore, homogeneous transformation matrix (HTM) is generally applied to model both operations and operational errors. For instance, HTM $H(k)$ is used to model the nominal setup at operation $k$. It transforms $X_0^j(j)$ from the nominal PCS (denoted as PCS0) to the nominal fixture coordinate system (FCS0). Since setup error could be induced by fixture error and datum error, we use HTMs $H(k)$ and $H(k)$ to denote the additional transformation of $X_j^j(k)$ in the FCS0 caused by fixture error and datum error, respectively.

To describe fixture error, the common 3-2-1 fixture locating scheme is adopted (Fig. 1). The fixture is represented by the positions of six locators in the FCS, i.e., $(f_1^x(k) f_2^y(k) f_3^z(k))^T, i = 1, \ldots, 6$. Not losing generality, the FCS0 is established with $f_1^x(k) = f_2^y(k) = f_3^z(k) = f_4^x(k) = f_5^y(k) = f_6^z(k) = 0$. The fixture error is described as deviations of locators, i.e., $\Delta f(k) = (\Delta f_1^x(k) \Delta f_2^y(k) \Delta f_3^z(k) \Delta f_4^x(k) \Delta f_5^y(k) \Delta f_6^z(k))^T$. Cai et al. [20] nicely presented the relationship between $\Delta f(k)$ and $H(k)$. The key results in [20] are summarized in Appendix A.

The datum error at operation $k$ is contained in the incoming workpiece $x(k-1)$. For the surfaces used as the primary, secondary, and tertiary datum, their errors are denoted as $x_1^j(k-1), x_2^j(k-1)$, and $x_3^j(k-1)$, respectively. The relationship between datum error and $H(k)$ is derived in Sec. 3 using the concept of EFE. The datum error is first converted to the equivalent amount of fixture locator errors (denoted as $\Delta d(k)$). Then the results in [20] can be directly applied to find $H(k)$ through $\Delta d(k)$.

The nominal cutting operation or the tool path can be modeled as $H(k)H(k)P^j(k)$, where $H(k)$ transforms a part surface from the FCS0 to the nominal machine tool coordinate system (MCS0). When deriving the results, we choose the MCS0 to be the same as the FCS0, i.e., $H(k)$ is identity matrix. Discussion is given in Sec. 3.3 when $H(k)$ is not identity matrix. We use $H(k)$ to represent the transformation of tool path (from nominal to the real one) caused by machine tool error. Only geometric errors of machine tool are considered [21].

As an example to show the form of HTM, $H_m(k)$ is given as

$$H_m(k) = \begin{pmatrix} \text{Rot}_m(k) & 0 & 0 \\ 0 & \text{Rot}_m(k) & 0 \\ 0 & 0 & \gamma_m(k) \end{pmatrix}$$

where $\text{Rot}_m(k)$ has the following form under small deviation assumption [20]

$$\text{Rot}_m(k) = \begin{pmatrix} 1 & -2\delta_{e3m} & 2\delta_{e2m} \\ 2\delta_{e1m} & 1 & -2\delta_{e2m} \\ -2\delta_{e1m} & 2\delta_{e2m} & 1 \end{pmatrix}$$

where $(\delta_{e1m}, \delta_{e2m}, \delta_{e3m})^T$ are deviations of Euler parameters, representing deviation of tool path orientation. $\text{Rot}_m(k)$ on the upper left corner of Eq. (4) transforms the orientation of surface, while the second $\text{Rot}_m(k)$ transforms the surface position: $(x_m^j(k) y_m^j(k) z_m^j(k))^T$ represent deviation of tool path in position; $\gamma_m(k)$ is the ratio of actual and ideal surface size. When $\gamma_m(k) = 1$, there is no size deviation due to machine tool error. Accordingly, we define machine tool error as $(\delta_{d1m}(k) \delta_{d2m}(k) - 1)^T$, where $\delta_{d1m}(x_m y_m z_m \delta_{e1m} \delta_{e2m} \delta_{e3m})^T$. The EFE due to machine tool is denoted as $\Delta m(k)$.

Notations $\delta_{d1m}(k)$ and $\delta_{d2m}(k)$ are also introduced for the parameters in $H_m(k)$ and $H(k)$. Since datum and fixture errors have no impact on the surface size, we have $\gamma_s(k) = \gamma_l(k) = 1$.

3 Variation Propagation Modeling Using Equivalent Fixture Error

We first introduce the concept of EFE. Using EFE, a variation propagation model is developed by grouping fixture, datum, and machine tool errors. Condition of error grouping is also discussed in this section.

3.1 Concept of Equivalent Fixture Error. The concept of EFE is based on the observation that datum and machine tool errors can generate the same error pattern on machined surfaces as fixture error. It can be illustrated with a two-dimensional block workpiece.

In Fig. 2, the dashed line block with surfaces $(X_0^1 \ X_0^2 \ X_0^3 \ X_0^0)$ is in its nominal setup position. Due to datum error occurring on surface $X_1$, the block has to be transformed to position $(X_1 X_2 X_3 X_4)$ (the solid line block) around the locating point $f_1$. The workpiece position transformation is described by HTM $H_1$.

$$H_1(k) = \begin{pmatrix} \text{Rot}_1(k) & 0 & 0 \\ 0 & \text{Rot}_1(k) & 0 \\ 0 & 0 & \gamma_1(k) \end{pmatrix}$$

where $\text{Rot}_1(k)$ has the following form under small deviation assumption [20]

$$\text{Rot}_1(k) = \begin{pmatrix} 1 & -2\delta_{e31} & 2\delta_{e21} \\ 2\delta_{e11} & 1 & -2\delta_{e21} \\ -2\delta_{e11} & 2\delta_{e21} & 1 \end{pmatrix}$$

where $(\delta_{e11}, \delta_{e21}, \delta_{e31})^T$ are deviations of Euler parameters, representing deviation of tool path orientation. $\text{Rot}_1(k)$ on the upper left corner of Eq. (4) transforms the orientation of surface, while the second $\text{Rot}_1(k)$ transforms the surface position: $(x_1^1(k) y_1^1(k) z_1^1(k))^T$ represent deviation of tool path in position; $\gamma_1(k)$ is the ratio of actual and ideal surface size. When $\gamma_1(k) = 1$, there is no size deviation due to machine tool error. Accordingly, we define machine tool error as $(\delta_{d11}(k) \gamma_1(k) - 1)^T$, where $\delta_{d11}(x_1 y_1 z_1 \delta_{e11} \delta_{e21} \delta_{e31})^T$. The EFE due to machine tool is denoted as $\Delta m(k)$.

Notations $\delta_{d11}(k)$ and $\delta_{d21}(k)$ are also introduced for the parameters in $H_1(k)$ and $H_1(k)$. Since datum and fixture errors have no impact on the surface size, we have $\gamma_1(k) = \gamma_1(k) = 1$.
The EFE due to datum error, denoted by $\Delta d(k)$, can be derived by finding the difference between actual $(H_d(k) - H_d(k)(X_{ik}^0(k-1) 1)^T)$ and nominal surface $(H_d(k)(X_{ik}^0(k-1) 1)^T)$, where $\{j \in \{I, II, III\}$, for $\Delta d(k)$ to be derived by Eq. (14). In Fig. 2, the equivalent fixture deviation is $\Delta d_1$ and $\Delta d_2$.

EFE due to machine tool error can be derived in a similar way. Figure 3(a) shows the machined surface $X_3$ due to machine tool error. The EFE transforms the workpiece from nominal position $(X_3^0, X_3^1, X_3^2)$ to dashed line position shown in Fig. 3(b). A nominal cutting operation can yield the same surface deviation as machine tool error does in Fig. 3(a). Therefore, the inverse of $H_m$ transforms $X_3$ to its nominal position $X_3^0$ in the FCS. The EFE due to machine tool error, denoted by $\Delta m(k) = \Delta m_1, \Delta m_2, \Delta m_3$ $(\Delta m_1, \Delta m_2, \Delta m_3)$, can be uniquely determined by the difference between $H_m^{-1}(k) H_d(k)(X_{ik}^0(k-1) 1)^T$ and $H_m(k)(X_{ik}^0(k-1) 1)^T$ at the locating point. $\{j \in \{I, II, III\}$, $\Delta m(k)$ is given by Eq. (12). In Fig. 3(b), the equivalent fixture locater deviation $\Delta m_1$, $\Delta m_2$ and $\Delta m_3$ is determined by difference between surfaces $X_3^0$ and $X_3$ at locating point 1 and 2. $\Delta m_3$ can be computed by the difference between surfaces $X_3^0$ and $X_3$ at locating point 3.

### 3.2 Variation Propagation Modeling Using EFE for Error Grouping

We use surface $x_j(k)$ to illustrate the derivation procedure (Fig. 4). It can be easily extended for part $x(k)$.

Step 1 models how feature quality is affected by faulty setup and cutting operation. Parameters $\delta q_j(k), \delta q_j(k)$, and $\delta q_m(k)$ in HTMs are intermediate variables linking $\Delta f(k), \Delta d(k)$ and $\Delta m(k)$ with feature deviation $x_j(k)$. Step 2 derives how fixture error $\Delta f(k), EFE \Delta d(k)$ and $\Delta m(k)$ affect $\delta q_j(k), \delta q_j(k)$, and $\delta q_m(k)$, respectively. Step 3 describes how errors from previous operation (datum error) affect $\Delta d(k)$.

**Step 1** After setup operation, the part surface can be represented by $H_p(k)(X_{ik}^0(k-1) 1)^T$. The machined surface $j$ is represented as $H_p(k)(X_{ik}^0(k-1) 1)^T$. After transforming the surface to the PCS $\{j\}$, the actual surface $X_{ik}(k)$ is

\[
X_{ik}(k) = (H_p(k)(X_{ik}^0(k-1) 1)^T \cdot (H_p(k)(X_{ik}^0(k-1) 1)^T)^T
\]

where $X_{ik}^0 = (v_{ik}^0, v_{ik}^0, v_{ik}^0, v_{ik}^0, r_{ijk}^0, r_{ijk}^0)$. By substituting Eqs. (4) and (5) into Eq. (6), we can compute the actual machined surface. After ignoring of higher order error terms, Eq. (6) can be rewritten as

\[
x_j(k) = A_{id}(k) - A_{id}(k) 0 0 18 \delta q_j(k) + x_j(k),
\]

where

\[
A_{id}(k) = \begin{pmatrix}
0 & 0 & 0 & -2v_{ik}^0 & -2v_{ik}^0 \\
0 & 0 & 0 & 2v_{ik}^0 & -2v_{ik}^0 \\
0 & 0 & 0 & -2v_{ik}^0 & 2v_{ik}^0 \\
-1 & 0 & 0 & 0 & -2v_{ik}^0 \\
0 & -1 & 0 & 2v_{ik}^0 & 0 \\
0 & 0 & -1 & 0 & 2v_{ik}^0
\end{pmatrix}
\]

and $\delta q_j(k) = (\delta q_j(k) - \delta q_j(k) q_j(k)) = (\delta q_j(k) - \delta q_j(k) q_j(k))$. $q_j(k)$ is modeling error for operation $k$. Index $k$ is omitted with $A_{id}(k), A_{id}(k)$ and $A_{id}(k)$.

The $\delta q_j(k)$ can be grouped because of $A_{id}(k) = A_{id}(k) = A_{id}(k)$.

Equation (7) can be rewritten as,

\[
x_j(k) = \left(A_{id}(k) 0 0 18 \delta q_j(k) 0 0 18 \delta q_j(k) \right) + ((\delta q_j(k) - \delta q_j(k)) - 1)^T + \delta q_j(k)
\]

where the dimension of $\delta q_j(k)$ is reduced from 19 to 7.

The expression for $A_{id}(k), A_{id}(k)$ and $A_{id}(k)$ in Eq. (7) is only given under the condition of $H_p(k) = I$. In Sec. 3.3, we will show that $A_{id}(k) = A_{id}(k) = A_{id}(k)$ and error grouping still hold if $H_p(k) \neq I$.

**Step 2** Relationship between $\delta q_j(k)$ and $\Delta f$ has been given as

\[
\delta q_j = - J^T \Phi E \Delta f
\]

by Cai et al. [20] (Refer to Appendix A for a brief summary of the result). By the concept of EFE, $\Delta f$ and $\Delta m$ are equivalent to $\Delta f$. Therefore, $\delta q_j(k)$ and $\delta q_m(k)$ can be determined accordingly by the same approach

\[
\delta q_j = - J^T \Phi E \Delta d
\]

\[
\delta q_m = - (- J^T \Phi E \Delta m) = J^T \Phi E \Delta m
\]

Since $H_m^{-1}$ (not $H_m$) transforms the workpiece from nominal position to its real position in the FCS [refer to Fig. 3(b)], we add minus sign before $- J^T \Phi E \Delta m$ in Eq. (10). It turns out that Jacobian matrix $J$ and orientation matrix $\Phi$ in Eqs. (9) and (10) are the same as those in Eq. (A1). Therefore, we still can group errors after substitute Eqs. (A1), (9), and (10) into Eq. (8)

\[
x_j(k) = B(k)(\Delta d(k) 0 T + B(k)(k) u(k) + \delta q_j(k)
\]

where

\[
B(k) = \left(- A_{id}(k) 0 0 18 \Phi 0 0 18 \Phi \right)
\]

is the input coefficient matrix linking errors at the current opera-
tion with feature deviation, rank \((-A_{\text{vec}}(k)\Gamma^{-1}(k)\Phi(k)E)\leq 5\), and \(u(k)=((\Delta f(k)+\Delta m(k))^T\gamma_m(k)-1)^T\).

As shown in Sec. 3.1, \(\Delta m(k)\) can be computed by finding the difference between actual surface and nominal surface at the locating points

\[
\Delta m_{i1}(k) = -2f_{i0}(k)\delta e_{i1m}(k) + 2f_{i3}(k)\delta e_{i2m}(k) - z_{m}(k), \quad i = 1, 2, 3,
\]

\[
\Delta m_{i2}(k) = 2f_{i0}(k)\delta e_{i1m}(k) - 2f_{i3}(k)\delta e_{i3m}(k) - y_{m}(k), \quad i = 4, 5,
\]

\[
\Delta m_{i3}(k) = -2f_{i0}(k)\delta e_{i2m}(k) + 2f_{i3}(k)\delta e_{i3m}(k) - x_{m}(k), \quad i = 6.
\]

(12)

**Step 3** EFE due to datum error can be determined by \(x_{k1}\), \(x_{k2}\) and \(x_{k3}\). If three datum surfaces are planar and \(f^T H_p = I\), the result can be concisely represented as

\[
\Delta d_{i1} = -f_{i0}\Delta u_{i1} - f_{i2}\Delta u_{i2} - \Delta p_{i1}, \quad i = 1, 2, 3,
\]

\[
\Delta d_{i2} = -f_{i0}\Delta u_{i2} - f_{i1}\Delta u_{i1} - \Delta p_{i2}, \quad i = 4, 5,
\]

\[
\Delta d_{i3} = -f_{i0}\Delta u_{i3} - f_{i1}\Delta u_{i1} - \Delta p_{i3}, \quad i = 6
\]

where operation index \(k\) is omitted. When \(f^T H_p \neq I\), \(\Delta d(k)\) becomes

\[
(\Delta d(k)) = \begin{pmatrix} \Psi & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} H_{22 \times 22} \begin{pmatrix} G & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{k1}(k-1) \\ x_{k2}(k-1) \\ x_{k3}(k-1) \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix}
\]

where matrix \(H\) transforms deviations of three datum surfaces from PCS\(^5\) to FCS\(^5\). It is defined as

\[
\begin{pmatrix} \tilde{f} R_p \Psi \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} x_p & y_p & z_p \end{pmatrix}^T \end{pmatrix} \begin{pmatrix} 0_{1 \times 21} \end{pmatrix} \begin{pmatrix} f_p & f_p & f_p & z_p \\ 0_{1 \times 1} \end{pmatrix}
\]

\[
\tilde{f} R_p = \text{diag}(\tilde{f} R_{p1} \gamma_{m1} \tilde{f} R_{p2} \gamma_{m2} \tilde{f} R_{p3} \gamma_{m3} \tilde{f} R_{p4} \gamma_{m4} \tilde{f} R_{p5})
\]

\(\tilde{f} R_{p}^T\gamma_m\tilde{f} R_{p}\) is the rotational block matrix in \(\tilde{f} H_p\). \(\tilde{f} R_{p}^T\gamma_m\tilde{f} R_{p}\) are translation parameters. Matrix

\[
\Psi = \begin{pmatrix} \Psi_1 & 0 & 0 \\ 0 & \Psi_2 & 0 \\ 0 & 0 & \Psi_3 \end{pmatrix}
\]

maps the deviation of workpiece to the EFE with

\[
\Psi_1 = \begin{pmatrix} f_{ix} & f_{iy} & 0 & 0 & 0 & 0 & 1 & 0 \\ f_{ix} & f_{iy} & 0 & 0 & 0 & 0 & 1 & 0 \\ f_{ix} & f_{iy} & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}
\]

\[
\Psi_2 = \begin{pmatrix} f_{ix} & f_{ix} & 0 & 0 & 0 & 0 & 1 & 0 \\ f_{ix} & f_{ix} & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}
\]

\[
\Psi_3 = (-0 f_{ix} f_{iy} f_{iz} 1 0 0 0). \]

Matrix \(G\) is introduced for computing deviation of orientation vector of datum surface under two conditions: (1) If all datum surfaces are planar: \(G=I\); (2) If \(X_i\) is plane, \(X_{k1}\) and \(X_{k3}\) are cylindrical holes, \(G\) can be obtained by differentiating \(v_{k1}\) with \(p_{i1}-p_{i3}\) and \(p_{i1}-p_{i3}\). The result is

\[
\begin{pmatrix} I_{7 \times 7} & 0 & 0 \\ 0 & G_{11} & G_{12} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}
\]

where

\[
G_{11} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}
\]

Substituting Eq. (14) into Eq. (11), state transition matrix \(A(k-1)\) can be obtained and we derive the variation propagation model for the surface \(j\) at operation \(k\). If we assemble the model for all the features and datum surfaces, the equation in the form of state space model shown in Model (1) can be obtained. The dimension of input vector \(u(k)\) is reduced from 13 to 7 because of error grouping. Thus the order of diagnosability matrix \(\Gamma^T \Gamma\) is greatly reduced. The dimension of output vector \(y(k)\) required to make \(\Gamma^T \Gamma\) full rank is also reduced.

**Remark**: The structure of Eq. (7) proves our claim in Sec. 1 that it is hard to conduct root cause identification using a previously developed model. It also reveals that fixture and machine tool cannot be distinguished without in-process measurements on either fixture locators or the machine tool at each operation.

### 3.3 Discussion for Error Grouping

In Sec. 3.2, the model derivation is based on the assumption that transformation matrix \(\tilde{f} H_m(k)\) is identity. In addition, the expression of \(A_p(k), A_p(k)\) and \(A_m(k)\) are given under the condition of \(\tilde{f} H_m(k)=I\). In this section, a necessary and sufficient condition for error grouping is discussed.

**Proposition (Condition on grouping variables)** The linear equation

\[
x = (x_1 x_2 \ldots x_n)^T \Gamma (u_1 u_2 \ldots u_m)^T
\]

where \(\Gamma=(\gamma_{ij})_{n \times m}, i=1,2,\ldots,n; j=1,2,\ldots,m, x_1,x_2,\ldots,x_n \) and \(u_1,u_2,\ldots,u_m\) are variables, can be grouped into the following form

\[
x = (p_1 p_2 \ldots p_n)^T u \quad \text{with} \quad \Gamma=\begin{pmatrix} k_1 & k_2 & \ldots & k_m \end{pmatrix} \quad \text{where} \quad k_1, k_2, \ldots, k_m \quad \text{are} \quad \begin{pmatrix} p_1 \ p_2 \ \ldots \ p_n \end{pmatrix}
\]

where \(p_i\) and \(k_j\) are certain coefficients, if and only if the rank of matrix \(\Gamma\) is one or zero.

**Corollary 1 (Condition on grouping vectors)** The equation

\[
x(k) = \Gamma(k) \Delta d + \Gamma(k) \Delta f + \Gamma(k) \Delta m
\]

\[
= \begin{pmatrix} \Gamma(k) \Delta d(k) \\ \Gamma(k) \Delta f(k) \\ \Gamma(k) \Delta m(k) \end{pmatrix}
\]

(17)

\[
\Gamma(k) = \begin{pmatrix} \Gamma(k) \Delta d(k) \\ \Gamma(k) \Delta f(k) \\ \Gamma(k) \Delta m(k) \end{pmatrix}
\]

where \(\Gamma(k), \Delta d(k), \Delta f(k), \Delta m(k)\) are constant coefficient matrices for variable vector \(\Delta d(k), \Delta f(k), \Delta m(k)\), respectively, can be
grouped into the form as \( x(k) = \Gamma(k)(k_1 \Delta d(k) + k_2 \Delta f(k) + k_3 \Delta m(k)) \), if and only if \( 1/k_1 \Gamma_f(k) = 1/k_2 \Gamma_m(k) = 1/k_3 \Gamma_m(k) \), where \( \Gamma(k) \) is a matrix, \( k_1, k_2, \) and \( k_3 \) are nonzero scalars.

In the above discussion, we assume the transformation matrix \( ^S H_p \) and \( ^F H_p \) to be identities. If three coordinate systems do not coincide with each other, the coefficient matrices for \( \Delta d \), \( \Delta f \), and \( \Delta m \) are still the same when \( ^S H_p \neq ^I S_{\times 8} \) and \( ^F H_M = ^I S_{\times 8} \). However, this is not true when \( ^S H_M \neq ^I S_{\times 8} \). We have the following conclusion.

**Corollary 2** MCS\(^0\) and FCS\(^0\) must coincide to perform error grouping in the proposed model. However, this requirement can be easily satisfied in the modeling stage. The proofs of the two corollaries are listed in Appendices B and C.

### 4 Case Studies

#### 4.1 Experimental Validation of EFE

We machined six blocks to validate the EFE model. The first three parts were cut with only datum error, while the rest were cut with only machine tool error. The datum error and machine tool error were set in such a way that \( \Delta d = \Delta m = (1.105 \ 0 \ 0 \ 0 \ 0 \ 0)^T \), i.e., their EFEs are the same based on Eqs. (12) and (13). Then we measured the machined surface and compared the surface orientation and position.

Figure 5 shows the specification of raw workpiece and fixture layout. Only top surface \( X \) was machined and its specification is \( X^0=(0 \ 0 \ 0 \ 20.32 \ 0)^T \). Using Eq. (6), the deviated surface \( X \) is predicted as \( (0 \ -0.0175 \ 0.9998 \ 0 \ 0 \ 18.88)^T \).

Table 1 shows the measurement of the machined surface. As can be seen, the discrepancies between two samples are very small. The measurement data are also comparable with the predicted results. Therefore, the experiment supported the EFE model.

#### 4.2 Modeling for Root Cause Identification

A machining process for a V-8 cylinder head is employed to illustrate the modeling procedure and the advantage of the modeling approach. The drawing of workpiece and the locating points are shown in Fig. 6.

The surfaces chosen are marked as \( X_1-X_8 \). \( X_1 \) is the exhaust face, while \( X_2 \) and \( X_3 \) are two cup plug holes on the \( X_1 \). \( X_4 \) is a spark plug tube hole and \( X_5 \) is a hole for the exhaust lash adjuster. \( X_6 \) and \( X_7 \) are two angle holes and the specifications are given in section plots \( S_1-S_1 \) and \( S_2-S_2 \). The center of \( X_7 \) is set to be the origin of nominal part coordinate system. Based on the data shown in Fig. 6, the specification of each machined surface is listed in Table 2.

The workpiece goes through two operations (Fig. 7): (1) mill \( X_1 \) and drill \( X_2 \) and \( X_3 \) using datum surfaces \( X_6 \), \( X_7 \), and \( X_5 \); and (2) drill \( X_4 \) and \( X_5 \) using datum surface \( X_1 \), \( X_2 \) and \( X_3 \).

Table 2 shows the measurements of the machined surface. As can be seen, the discrepancies between two samples are very small. The measurement data are also comparable with the predicted results. Therefore, the experiment supported the EFE model.

**Table 1 Measurement results (under PCS\(^0\))**

<table>
<thead>
<tr>
<th>( X )</th>
<th>( v_x )</th>
<th>( v_y )</th>
<th>( v_z )</th>
<th>( p_x )</th>
<th>( p_y )</th>
<th>( p_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample 1 (datum error)</td>
<td>0</td>
<td>-0.0174</td>
<td>0.9998</td>
<td>0</td>
<td>0</td>
<td>18.880</td>
</tr>
<tr>
<td>Sample 2 (machine tool error)</td>
<td>0</td>
<td>-0.0174</td>
<td>0.9998</td>
<td>0</td>
<td>0</td>
<td>18.882</td>
</tr>
<tr>
<td>Sample 1 (datum error)</td>
<td>0</td>
<td>-0.0174</td>
<td>0.9998</td>
<td>0</td>
<td>0</td>
<td>18.881</td>
</tr>
<tr>
<td>Sample 2 (machine tool error)</td>
<td>0</td>
<td>-0.0174</td>
<td>0.9998</td>
<td>0</td>
<td>0</td>
<td>18.880</td>
</tr>
<tr>
<td>Sample 1 (datum error)</td>
<td>0</td>
<td>-0.0173</td>
<td>0.9999</td>
<td>0</td>
<td>0</td>
<td>18.884</td>
</tr>
<tr>
<td>Sample 2 (machine tool error)</td>
<td>0</td>
<td>-0.0163</td>
<td>0.9999</td>
<td>0</td>
<td>0</td>
<td>18.887</td>
</tr>
</tbody>
</table>
each operation. However, we have shown in Eq. (11) that the rank of block matrix $-A_{jd}^{-1}(k) \Phi(k)E$ in $B_{j}(k)$ does not exceed 5. More features information is needed to identify all the errors. Therefore, the number of features identifying errors for each operation should be no less than 3. In this case study where only 2 operations are considered, the total amount of measured features should not be less than $3 \times 2 = 6$ even if the purpose is to identify whether errors occur in the process.

Using Eqs. (11) and (14), we calculate $A_{j}(k)$ and $B_{j}(k)$, based on which the model in the grouped form is formulated as follows.

Operation 1: Because the first operation only mills $X_1$ and drills $X_2$ and $X_3$, input matrices for feature 4 and 5 are zero. The results are

<table>
<thead>
<tr>
<th>Feature Component</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
<th>$X_6$</th>
<th>$X_7$</th>
<th>$X_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_x(k)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$v_y(k)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.43</td>
<td>0.28</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$v_z(k)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.90</td>
<td>-0.96</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$p_x(k)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$p_y(k)$</td>
<td>131</td>
<td>131</td>
<td>131</td>
<td>52.69</td>
<td>44.41</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$p_z(k)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$r(k)$</td>
<td>0</td>
<td>7.5</td>
<td>7.5</td>
<td>4.6</td>
<td>16.92</td>
<td>0</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 2 Machined features specification

<table>
<thead>
<tr>
<th>Feature Component</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
<th>$X_6$</th>
<th>$X_7$</th>
<th>$X_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_x(k)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$v_y(k)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.43</td>
<td>0.28</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$v_z(k)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.90</td>
<td>-0.96</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$p_x(k)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$p_y(k)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$p_z(k)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$r(k)$</td>
<td>0</td>
<td>7.5</td>
<td>7.5</td>
<td>4.6</td>
<td>16.92</td>
<td>0</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 3 Coordinates of locating points on the primary datum surfaces (Unit: mm)

<table>
<thead>
<tr>
<th>Operation</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-7, 109, 0)</td>
<td>(407, 109, 0)</td>
<td>(200, -11, 0)</td>
</tr>
</tbody>
</table>
The state equation for operation $k$ can be assembled as

$$
\begin{pmatrix}
    x_1(1) \\
    x_2(1) \\
    x_3(1) \\
    x_4(1) \\
    x_5(1)
\end{pmatrix}
= \begin{pmatrix}
    I_{7x7} & 0 & 0 & 0 & 0 \\
    0 & I_{7x7} & 0 & 0 & 0 \\
    0 & 0 & I_{7x7} & 0 & 0 \\
    0 & 0 & 0 & I_{7x7} & 0 \\
    0 & 0 & 0 & 0 & I_{7x7}
\end{pmatrix}
\begin{pmatrix}
    x(1)^T \\
    1
\end{pmatrix}
+ \begin{pmatrix}
    B_1(1) \\
    B_2(1) \\
    B_3(1) \\
    B_4(1) \\
    B_5(1)
\end{pmatrix}
\begin{pmatrix}
    \Delta f_{1x1}(1) + \Delta m_{1x1}(1) \\
    \Delta f_{2x1}(1) + \Delta m_{2x1}(1) \\
    \Delta f_{3x1}(1) + \Delta m_{3x1}(1) \\
    \Delta f_{4x1}(1) + \Delta m_{4x1}(1) \\
    \Delta f_{5x1}(1) + \Delta m_{5x1}(1)
\end{pmatrix}
+ \begin{pmatrix}
    \bar{g}(1) \\
    0
\end{pmatrix},
$$

where identity block matrix in $A(0)$ represents that the corresponding features have not been machined. Since HTM is used to derive $\Delta d(k)$ as shown in Eq. (14), the dimension of state vector has to be increased by using “1” as the last entry, i.e., $(x^T(1))^T$. Zeros in the last row of the model are introduced to make the matrix dimension consistent.

Operation 2: Since $\tilde{H}_p(2) \neq I$, the expression of $A_{jd}(k)$ presented in Eq. (7) does not apply for the second operation. However, according to corollary 2 in Sec. 3.3, we can still derive $A(1)$ and $B(2)$ by substituting nonidentity matrix $\tilde{H}_p$ in Eq. (6), followed by the same procedure for deriving Eqs. (7), (8), and (11).

$$
B_1(2) = \begin{pmatrix}
    0 & 0.001 & -0.001 & -0.0023 & 0.0023 & 0 \\
    -0.0075 & 0.0038 & 0.0038 & 0 & 0 \\
    0.0036 & -0.0018 & 0.018 & 0 & 0 \\
    0 & 0.1892 & 0.1892 & 0.2025 & -0.2025 & -1 \\
    1.5833 & -0.6081 & 0.0248 & 0 & 0 \\
    -0.6526 & 0.3263 & 0.3263 & -0.1725 & -0.8275 & 0
\end{pmatrix}_{6 \times 1}
\begin{pmatrix}
    0 \\
    0.007 \\
    -0.007 \\
    -0.0024 \\
    0 \\
    0
\end{pmatrix}_{6 \times 1}
+ \begin{pmatrix}
    0.6264 & 0.5961 & -0.2203 & 0 & 0 \\
    0.7216 & 0.3608 & 0.3608 & -0.9225 & -0.0775 & 0
\end{pmatrix}_{1 \times 6}
+ \begin{pmatrix}
    1.5833 & -0.6081 & 0.0248 \\
    0.6264 & 0.5961 & -0.2203 \\
    -0.7216 & 0.3608 & 0.3608 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix}_{1 \times 6}
+ \begin{pmatrix}
    0 & 0.007 & -0.007 & 0.0024 \\
    0.008 & -0.004 & -0.004 & 0 & 0 \\
    0.0023 & -0.0011 & -0.0011 & 0 & 0 \\
    0 & 0.2092 & 0.2092 & -0.0852 & 0.0852 & -1 \\
    0.6264 & 0.5961 & -0.2203 & 0 & 0 \\
    -0.7216 & 0.3608 & 0.3608 & -0.9225 & -0.0775 & 0
\end{pmatrix}_{1 \times 6}
$$

$B_2(2) = B_1(2) = \text{diag}(0_{6 \times 7}, 7.5)$.

Since datum error is generated by operation 1, state transition matrix must be calculated. By Eqs. (11) and (14), rotational deviation of the surface caused by datum errors can be expressed by

$$
-A_{jd}(k)J^{-1}(k)\Phi(k)E\Psi^TR_pG
\begin{pmatrix}
    x_1(1) \\
    x_2(1) \\
    x_3(1)
\end{pmatrix},
$$

where $j=4, 5$. For the convenience of displaying results, we denote $(A_{j1} A_{j2} A_{j3})=-A_{jd}(k)J^{-1}(k)\Phi(k)E\Psi^TR_pG$. The results are

$$
A_{j1} = \begin{pmatrix}
    0.4305 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & -0.9026 & 0 & 0 & 0 & 0 \\
    0 & 0.4305 & 0 & 0 & 0 & 0 & 0 \\
    -78.31 & 0 & 0 & 0 & 0 & 0 & 0 \\
    -331 & 0 & 81 & 0 & 0 & 0 & 0 \\
    0 & -78.31 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}_{7 \times 7}
\begin{pmatrix}
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}_{7 \times 7}
+ \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}_{7 \times 7}
$$

$A_{j2} = \begin{pmatrix}
    0 & 0 & 0 & 0.9026 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & -331 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}_{7 \times 7}
$$

$A_{j3} = \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}_{7 \times 7}$.
The translational deviation of surface can be calculated by

$$-A_{j4}(k) F(k) \Phi(k) \Psi(k_1 x_1 f_{x p} f_{y p} f_{z p} 0_{1 \times 4} f_{x p} f_{y p} f_{z p} 0_{1 \times 4} f_{x p} f_{y p} f_{z p} 0)^T.$$ We denote this expression as a column vector $A_{j4}$. The calculation results are $A_{j4} = 0_{1 \times 3} - 19.25 - 131 - 81.25 0^T$. The state equation can be assembled as

$$\begin{bmatrix}
\Delta f_4(2) + \Delta m_{14}(2) \\
\Delta f_2(2) + \Delta m_{24}(2) \\
\Delta f_{35}(2) + \Delta m_{35}(2) \\
\Delta f_{56}(2) + \Delta m_{56}(2)
\end{bmatrix} +
\begin{bmatrix}
\zeta(1) \\
\zeta(2)
\end{bmatrix}.$$ 

Solving the state equation for two operations, the model for root cause identification is given by

$$y = \begin{pmatrix} y(1) \\ y(2) \end{pmatrix}^T = C \begin{pmatrix} B(1) & 0 \\ A(1) B(1) & B(2) \end{pmatrix} \begin{pmatrix} u(1) \\ u(2) \end{pmatrix}^T + \begin{pmatrix} \zeta(1) \\ \zeta(2) \end{pmatrix}.$$ 

Output matrix $C$ is determined by selection of measured features. An optimized selection of measured features for root cause identification must maximize the rank of matrix $G$, while minimizing the number of rows in matrix $C$, i.e., the minimum number of components in vector $y$. In this example, the number of errors to be determined is 12 and the minimum number of feature components to be measured should be 12. Each feature component is selected as one entry in vector $x_j(1)$, e.g., $y_j(1)$ in $x_j(1)$ can be chosen as a feature component. The entry “1” appears at most once in each row of feature selection matrix $C$. The position of “1” is determined by nonzero entry in

$$\begin{pmatrix} B(1) & 0 \\ A(1) B(1) & B(2) \end{pmatrix}.$$ 

For this case study, four features are selected and the output matrix $C$ is chosen as

$$C_{12x72} = \begin{pmatrix} 0 & C_1 & C_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$C_1 = \begin{pmatrix} 1 & 0 & 0_{4 \times 1} \\ 0_{4 \times 1} & 0_{3 \times 1} & I_{4 \times 4} & 0_{6 \times 1} & 0_{4 \times 4} \end{pmatrix}, C_2 = \begin{pmatrix} 0_{5 \times 5} & 0_{5 \times 1} \\ 0_{5 \times 1} & 0_{6 \times 1} \\ 0_{6 \times 1} & 1 \end{pmatrix}.$$ 

The size of other zero block matrices in $C$ is $7 \times 7$. After removing the zero rows in $u$ and corresponding columns in

$$C \begin{pmatrix} B(1) & 0 \\ A(1) B(1) & B(2) \end{pmatrix},$$

we get equation

$$y = \begin{pmatrix} y(1) \\ y(2) \end{pmatrix} = \begin{pmatrix} \Delta f(1) + \Delta m(1) \\ \Delta f(1) + \Delta m(1) \end{pmatrix} + \varepsilon$$

for diagnosis of errors that occur at each operation, where $\varepsilon$ is the noise terms composed of $\zeta(1)$ and $\zeta(2)$ in the first and second operations and
It can be observed that the rank of $\Gamma$ is 12. The least square estimation can thus be performed. Therefore, measuring 4 features makes it possible to identify 12 error components. Only 12 components in quality characteristic $v$ are needed for identifying if there are errors. The proposed approach identifies the location of the root cause without having to find out every potential error. Compared with quality characteristic components (at least 24) and 6 features measured for the previous model [11-13,17], reduction on the model dimension and measurements by the proposed approach are significant. If the fixture and machine tool errors should be further distinguished, the strategy of sequential root cause identification suggests that additional in-process measurement only needs to be taken on the faulty (equivalent) locator(s). Therefore, the proposed strategy generally requires less features and in-process measurement for root cause identification.

5 Conclusions

The paper presents a modeling procedure that facilitates root cause identification and measurement strategy. This is achieved by directly modeling the process physics regarding how fixture, and datum, and machine tool errors generate the same pattern on part features. Through the EFE model, datum error and machine tool error can be grouped with fixture error. As a result, the dimension of model inputs is significantly reduced compared with previous modeling methodology. In addition, root cause identification can be conducted in a sequential manner: first, if any error is identified, further measurement will be conducted to distinguish among three types of errors occurred in the problematic locator(s).

The feasibility of error grouping is discussed. It is shown that the symmetry of HTM in the infinitesimal analysis is the key factor for error grouping. We discuss the effect of coordinate transformation that may possibly violate the symmetry in HTM multiplication. The results indicate that HTM between the PCS and the FCS does not affect the symmetry in HTM multiplication. This grouping approach requires merging the MCS and the FCS during modeling to satisfy the condition of grouping. The requirement can easily be satisfied in the modeling stage.

The case studies demonstrated the validity of the EFE model, modeling procedure, and its implementation in measurement reduction.

Since datum, fixture, and machine tool errors might cancel each other in the process, error information in the system is potentially concealed. Although the effect of error cancellation exists, modeling with grouping approach is shown to be an efficient way to assist diagnosis and measurement reduction. Future research can be further studied on: (1) procedure for sequential root cause identification; (2) the effect of error cancellation on different features; and (3) feature selection to distinguish among errors when errors are cancelled on certain features.

Acknowledgment

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Appendix A: Infinitesimal Analysis of Workpiece Deviation due to Fixture Error

If there are small deviations on these six locators as $(f_{1x}, f_{2x}, f_{3x}, f_{4y}, f_{5y}, f_{6y})$, the change of orientation and position of rigid workpiece in the three-dimensional space can be analyzed by [20],

$$\delta x = J^{-1}\Phi \Delta f$$  \hspace{1cm} (A1)

where for prismatic workpiece, Jacobian Matrix $J$ is

$$J = \begin{pmatrix} v_{tx} & v_{ty} & v_{tz} & -v_{tx} & -v_{ty} & -v_{tz} \\ -v_{tx} & -v_{ty} & -v_{tz} & v_{tx} & v_{ty} & v_{tz} \\ -v_{tx} & -v_{ty} & -v_{tz} & v_{tx} & v_{ty} & v_{tz} \\ -v_{tx} & -v_{ty} & -v_{tz} & v_{tx} & v_{ty} & v_{tz} \\ -v_{tx} & -v_{ty} & -v_{tz} & v_{tx} & v_{ty} & v_{tz} \\ -v_{tx} & -v_{ty} & -v_{tz} & v_{tx} & v_{ty} & v_{tz} \end{pmatrix} \begin{pmatrix} f_{1x} & v_{tb1} & f_{1y} & v_{tb2} & f_{1z} & v_{tb3} \\ f_{2x} & v_{tb1} & f_{2y} & v_{tb2} & f_{2z} & v_{tb3} \\ f_{3x} & v_{tb1} & f_{3y} & v_{tb2} & f_{3z} & v_{tb3} \\ f_{4x} & v_{tb1} & f_{4y} & v_{tb2} & f_{4z} & v_{tb3} \\ f_{5x} & v_{tb1} & f_{5y} & v_{tb2} & f_{5z} & v_{tb3} \\ f_{6x} & v_{tb1} & f_{6y} & v_{tb2} & f_{6z} & v_{tb3} \end{pmatrix}$$  \hspace{1cm} (A2)

where $v_j=(v_{jx}, v_{jy}, v_{jz})^T$ is the orientation vector of datum surface $j$ and the index $k$ is dropped in the equations in the Appendix. The Jacobian matrix $J$ is definitely full rank because the workpiece is deterministically located. The inverse of Jacobian therefore exists. Matrix $\Phi$ is
When it is clear in the text, index $k$ is dropped in the above equation. $E$ is an $18 \times 6$ matrix,

$$
\phi = \begin{pmatrix}
E_1 & 0 & 0 & 0 & 0 & 0 \\
0 & E_2 & 0 & 0 & 0 & 0 \\
0 & 0 & E_3 & 0 & 0 & 0 \\
0 & 0 & 0 & E_2 & 0 & 0 \\
0 & 0 & 0 & 0 & E_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

that is,

$$
\begin{pmatrix}
E_1 & 0 & 0 & 0 & 0 & 0 \\
0 & E_2 & 0 & 0 & 0 & 0 \\
0 & 0 & E_3 & 0 & 0 & 0 \\
0 & 0 & 0 & E_2 & 0 & 0 \\
0 & 0 & 0 & 0 & E_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}^{18 \times 6}
$$

where $E_1 = (0 \ 0 \ 1)^T$, $E_2 = (0 \ 1 \ 0)^T$, and $E_3 = (1 \ 0 \ 0)^T$.

**Appendix B: Proof for Proposition and Corollary 1**

**Proof** If the variables $u_1, u_2, \ldots, u_n$ can be grouped to Eq. (16), we can expand Eqs. (15) and (16) and make them equal. Then we get $k, p_i = 0$. Substituting it into $\Gamma$ yields

$$
\Gamma = \begin{pmatrix}
k_1p_1 & k_2p_2 & \ldots & k_np_n \\
\end{pmatrix}
$$

whose rank is not larger than 1. On the other hand, if $\text{rank}(H)$ is less than 1, there exists at most one row that is linearly independent. The conclusion is obvious.

The proof for corollary 1 can be derived by applying the condition on grouping variables for corresponding entries in $\Delta d$, $\Delta f$, and $\Delta m$. In our study, the coefficient matrices of $\Delta d$, $\Delta f$, and $\Delta m$ are the same, [see Eqs. (A1), (9), and (10)], which satisfies the sufficient condition for grouping.

**Appendix C: Proof for Corollary 2**

**Proof** This can be proved by substituting Eqs. (4) and (5) into the expression $(X_f X'_f)^T \hat{H}_{ij} \hat{H}_{ij}^{-1} \hat{H}_{ij} \hat{H}_{ij}^{-1} \hat{H}_{ij} - \hat{H}_{ij} \hat{H}_{ij}^{-1} \hat{H}_{ij} \hat{H}_{ij}^{-1} \hat{H}_{ij}^{-1} \hat{H}_{ij} = (X_f X'_f)^T)$ and conducting a lengthy computation. It can be found that equalities among the coefficient matrices are determined by the symmetry of matrix $H_{ij}^{-1} H_{ij}^{-1} H_{ij}^{-1} H_{ij}^{-1} H_{ij}^{-1} H_{ij}^{-1}$. Since $H_i$, $H_j$, and $H_m$ are skew-symmetric, $H_{ij}^{-1} H_{ij}^{-1} H_{ij}^{-1} H_{ij}^{-1} H_{ij}^{-1} H_{ij}^{-1}$ is also skew-symmetric if $H_{ij}^{-1} = I_{ij}$. Non-identity matrix $H_{ij}$ can affect the symmetry of $H_{ij}^{-1} H_{ij}^{-1} H_{ij}^{-1} H_{ij}^{-1} H_{ij}^{-1} H_{ij}^{-1}$, which yields different coefficient matrices for $\Delta d$, $\Delta f$, and $\Delta m$. Therefore, the MSC and the FCS must coincide with each other for the proposed grouping method.

**References**