Chapter 2 Macromechanical Analysis of a Lamina
Tsai-Hill Failure Theory

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Courtesy of the Textbook
Mechanics of Composite Materials by Kaw
The failure theories are generally based on the normal and shear strengths of a unidirectional lamina.

In the case of a unidirectional lamina, the five strength parameters are:

- Longitudinal tensile strength $\left( \sigma_{1}^{T} \right)_{ult}$
- Longitudinal compressive strength $\left( \sigma_{1}^{C} \right)_{ult}$
- Transverse tensile strength $\left( \sigma_{2}^{T} \right)_{ult}$
- Transverse compressive strength $\left( \sigma_{2}^{C} \right)_{ult}$
- In-plane shear strength $\left( \tau_{12} \right)_{ult}$
Based on the distortion energy theory, Tsai and Hill proposed that a lamina has failed if

\[
(G_2 + G_3)\sigma_1^2 + (G_1 + G_3)\sigma_2^2 + (G_1 + G_2)\sigma_3^2 - 2G_3\sigma_1\sigma_2 - 2G_2\sigma_1\sigma_3 \\
- 2G_1\sigma_2\sigma_3 + 2G_4\tau_{23}^2 + 2G_5\tau_{13}^2 + 2G_6\tau_{12}^2 < 1
\]

- This theory is based on the interaction failure theory.
- The components $G_1$ thru $G_6$ of the strength criteria depend on the strengths of a unidirectional lamina.
Components of Tsai-Hill Failure Theory

\[
\left(G_2 + G_3\right)\sigma_1^2 + \left(G_1 + G_3\right)\sigma_2^2 + \left(G_1 + G_2\right)\sigma_3^2 - 2G_3\sigma_1\sigma_2 - 2G_2\sigma_1\sigma_3 \\
- 2G_1\sigma_2\sigma_3 + 2G_4\tau_{23}^2 + 2G_5\tau_{13}^2 + 2G_6\tau_{12}^2 < 1
\]

Apply \(\sigma_1 = (\sigma_1^T)_{ult}\) to a unidirectional lamina, then the lamina will fail. Hence, Equation reduces to

\[
\left(G_2 + G_3\right)(\sigma_1^T)_{ult}^2 = 1
\]
Components of Tsai-Hill Failure Theory

\[
(G_2 + G_3)\sigma_1^2 + (G_1 + G_3)\sigma_2^2 + (G_1 + G_2)\sigma_3^2 - 2G_3\sigma_1\sigma_2 - 2G_2\sigma_1\sigma_3 - 2G_1\sigma_2\sigma_3 + 2G_4\tau_{23}^2 + 2G_5\tau_{13}^2 + 2G_6\tau_{12}^2 < 1
\]

Apply \(\sigma_2 = (\sigma_2^T)_{ult}\), to a unidirectional lamina, then the lamina will fail. Hence, Equation reduces to

\[
(G_1 + G_3)(\sigma_2^T)_{ult}^2 = 1
\]
Apply $\sigma_3 = (\sigma_2^T)_{ult}$, to a unidirectional lamina, and assuming that the normal tensile failure strength is the same in direction (2) and (3), then the lamina will fail. Hence, Equation reduces to

$$
\begin{align*}
\left( G_2 + G_3 \right) \sigma_1^2 + \left( G_1 + G_3 \right) \sigma_2^2 + \left( G_1 + G_2 \right) \sigma_3^2 - 2 G_3 \sigma_1 \sigma_2 - 2 G_2 \sigma_1 \sigma_3 \\
- 2 G_1 \sigma_2 \sigma_3 + 2 G_4 \tau_{23}^2 + 2 G_5 \tau_{13}^2 + 2 G_6 \tau_{12}^2 < 1
\end{align*}
$$

Apply $\sigma_3 = (\sigma_2^T)_{ult}$, to a unidirectional lamina, and assuming that the normal tensile failure strength is the same in direction (2) and (3), then the lamina will fail. Hence, Equation reduces to

$$
\left( G_1 + G_2 \right) \left( \sigma_2^T \right)_{ult}^2 = 1
$$
Components of Tsai-Hill Failure Theory

\[ \left( G_2 + G_3 \right) \sigma_1^2 + \left( G_1 + G_3 \right) \sigma_2^2 + \left( G_1 + G_2 \right) \sigma_3^2 - 2 G_3 \sigma_1 \sigma_2 - 2 G_2 \sigma_1 \sigma_3 - 2 G_1 \sigma_2 \sigma_3 + 2 G_4 \tau_{23}^2 + 2 G_5 \tau_{13}^2 + 2 G_6 \tau_{12}^2 < 1 \]

Apply \( \tau_{12} = (\tau_{12})_{ult} \) to a unidirectional lamina, then the lamina will fail. Hence, Equation reduces to

\[ 2 G_6 (\tau_{12})_{ult}^2 = 1 \]
Components of Tsai-Hill Failure Theory

\[
(G_2 + G_3)(\sigma_1^T)_{ult}^2 = 1
\]

\[
(G_1 + G_3)(\sigma_2^T)_{ult}^2 = 1
\]

\[
(G_1 + G_2)(\sigma_2^T)_{ult}^2 = 1
\]

\[
2G_6(\tau_{12})_{ult}^2 = 1
\]

\[
G_1 = \frac{1}{2} \left( \frac{2}{(\sigma_2^T)_{ult}^2} - \frac{1}{(\sigma_1^T)_{ult}^2} \right)
\]

\[
G_2 = \frac{1}{2} \left( \frac{1}{(\sigma_1^T)_{ult}^2} \right)
\]

\[
G_3 = \frac{1}{2} \left( \frac{1}{(\sigma_2^T)_{ult}^2} \right)
\]

\[
G_6 = \frac{1}{2} \left( \frac{1}{(\tau_{12})_{ult}^2} \right)
\]
Because the unidirectional lamina is assumed to be under plane stress - that is, \( \sigma_3 = \tau_{31} = \tau_{23} = 0 \),

\[
(G_2 + G_3)\sigma_1^2 + (G_1 + G_3)\sigma_2^2 + (G_1 + G_2)\sigma_3^2 - 2G_3\sigma_1\sigma_2 - 2G_2\sigma_1\sigma_3 - 2G_1\sigma_2\sigma_3 + 2G_4\tau_{23}^2 + 2G_5\tau_{13}^2 + 2G_6\tau_{12}^2 < 1
\]

\[
\left[\frac{\sigma_1}{(\sigma_1^T)_{ult}}\right]^2 - \left[\frac{\sigma_1\sigma_2}{(\sigma_1^T)^2_{ult}}\right] + \left[\frac{\sigma_2}{(\sigma_2^T)_{ult}}\right]^2 + \left[\frac{\tau_{12}}{(\tau_{12})_{ult}}\right]^2 < 1
\]
Unlike the Maximum Strain and Maximum Stress Failure Theories, the Tsai-Hill failure theory considers the interaction among the three unidirectional lamina strength parameter.

The Tsai-Hill Failure Theory does not distinguish between the compressive and tensile strengths in its equation. This can result in underestimation of the maximum loads that can be applied when compared to other failure theories.

Tsai-Hill Failure Theory underestimates the failure stress because the transverse strength of a unidirectional lamina is generally much less than its transverse compressive strength.
Find the maximum value of $S > 0$ if a stress of $\sigma_x = 2S, \sigma_y = -3S$, and $\tau_{xy} = 4S$ is applied to a $60^\circ$ lamina of Graphite/Epoxy. Use Tsai-Hill Failure Theory. Use properties of a unidirectional Graphite/Epoxy lamina given in Table 2.1 of the textbook *Mechanics of Composite Materials by Autar Kaw*. 
From Example 2.13,

\[ \sigma_1 = 1.714 S, \]
\[ \sigma_2 = -2.714 S, \]
\[ \tau_{12} = -4.165 S. \]

Using the Tsai-Hill failure theory from Equation (2.150),

\[ \left( \frac{1.714S}{1500 \times 10^6} \right)^2 - \left( \frac{1.714S}{1500 \times 10^6} \right) \left( \frac{-2.714S}{1500 \times 10^6} \right) + \left( \frac{-2.714S}{40 \times 10^6} \right)^2 + \left( \frac{-4.165S}{68 \times 10^6} \right)^2 < 1 \]

\[ S < 10.94 \text{ MPa} \]

a) The Tsai-Hill failure theory considers the interaction between the three unidirectional lamina strength parameters, unlike the Maximum Strain and Maximum Stress failure theories.

b) The Tsai-Hill failure theory does not distinguish between the compressive and tensile strengths in its equations. This can result in underestimation of the maximum loads that can be applied when compared to other failure theories. For the load of \( \sigma_x = 2 \text{ MPa}, \sigma_y = -3 \text{ MPa} \) and \( \tau_{xy} = 4 \text{ MPa} \) as found in Examples 2.15, 2.17 and 2.18, the strength ratios are given by
Example 2.18

\[ SR = 10.94 \] (Tsai-Hill failure theory),

\[ = 16.33 \] (Maximum Stress failure theory),

\[ = 16.33 \] (Maximum Strain failure theory).

Tsai-Hill failure theory underestimates the failure stress because the transverse tensile strength of a unidirectional lamina is generally much less than its transverse compressive strength. The compressive strengths are not used in the Tsai-Hill failure theory. The Tsai-Hill failure theory can be modified to use corresponding strengths, tensile or compressive, in the failure theory as follows

\[
\left[ \frac{\sigma_1}{X_1} \right]^2 - \left[ \frac{\sigma_1}{X_2} \left( \frac{\sigma_2}{Y} \right) \right] + \left[ \frac{\tau_{12}}{S} \right]^2 < 1
\]  

(2.151)

where

\[ X_1 = (\sigma_1^T)_{ult} \text{ if } \sigma_1 > 0 \]

\[ = (\sigma_1^C)_{ult} \text{ if } \sigma_1 < 0 \]

\[ X_2 = (\sigma_2^T)_{ult} \text{ if } \sigma_2 > 0 \]

\[ = (\sigma_2^C)_{ult} \text{ if } \sigma_2 < 0 \]
\[ Y = (\sigma_2^T)_{ult} \text{ if } \sigma_2 > 0 \]
\[ = (\sigma_2^E)_{ult} \text{ if } \sigma_2 < 0 \]
\[ S = (\tau_{12})_{ult}. \]

For Example 2.18, the modified Tsai-Hill failure theory given by Equation (2.151) now gives

\[ \left( \frac{1.714\sigma}{1500 \times 10^6} \right)^2 \left( \frac{1.714\sigma}{1500 \times 10^6} \right) \left( \frac{-2.714\sigma}{1500 \times 10^6} \right) + \left( \frac{-2.714\sigma}{246 \times 10^6} \right)^2 + \left( \frac{-4.165\sigma}{68 \times 10^6} \right)^2 < 1 \]

\[ \sigma < 16.06 \text{ MPa}. \]

which implies that the strength ratio is \( SR = 16.06 \) (modified Tsai-Hill failure theory)

This value is closer to the values obtained using Maximum Stress and Maximum Strain failure theories.

c) The Tsai-Hill failure theory is a unified theory and hence does not give the mode of failure like the Maximum Stress and Maximum Strain failure theories.

However, you can make a reasonable guess of the failure mode by calculating \( |\sigma_i / (\sigma_1^T)_{ult}|, |\sigma_2 / (\sigma_2^T)_{ult}| \text{ and } |\tau_{12} / (\tau_{12})_{ult}| \). The maximum of these three values gives the associated mode of failure. In the modified Tsai-Hill failure theory, calculate the maximum of

\[ |\sigma_i / X_1|, |\sigma_2 / Y| \text{ and } |\tau_{12} / S| \] for the associated mode of failure.
Modified Tsai-Hill Failure Theory

\[
\left[ \frac{\sigma_1}{X_1} \right]^2 - \left[ \left( \frac{\sigma_1}{X_2} \right) \left( \frac{\sigma_2}{X_2} \right) \right] + \left[ \frac{\sigma_2}{Y} \right]^2 + \left[ \frac{\tau_{12}}{S} \right]^2 < 1
\]

\[
X_1 = \left( \sigma^T_1 \right)_{ult}, \text{if } \sigma_1 > 0
\]
\[
= \left( \sigma^C_1 \right)_{ult}, \text{if } \sigma_1 < 0
\]

\[
X_2 = \left( \sigma^T_1 \right)_{ult}, \text{if } \sigma_2 > 0
\]
\[
= \left( \sigma^C_1 \right)_{ult}, \text{if } \sigma_2 < 0
\]

\[
Y = \left( \sigma^T_2 \right)_{ult}, \text{if } \sigma_2 > 0
\]
\[
= \left( \sigma^C_2 \right)_{ult}, \text{if } \sigma_2 < 0
\]

\[
S = \left( \tau_{12} \right)_{ult}
\]