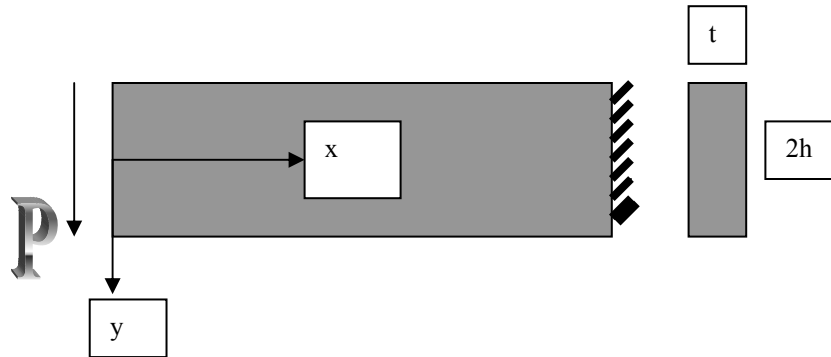


**Example of Solving a Two Dimensional Elasticity**

**Problem Using Airy Stress Function**



Starting from

$$\phi = d_1x^4 + d_2x^3y + d_3x^2y^2 + d_4xy^3 + d_5y^4 + c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3 + b_1x^2 + b_2xy + b_3y^2 + a_1x + a_2y + a_3$$

find the stresses,  $\sigma_{xx}$ ,  $\sigma_{yy}$  and  $\tau_{xy}$  for the following beam problem. Assume plane stress and satisfy the boundary conditions on the left and right edges at least globally.

- Write the boundary conditions locally on top and bottom edge. Write the boundary conditions globally on the left and right edges.
- Find the expressions for the stresses in terms of the constants.
- Apply all boundary conditions and  $\nabla^4\phi = 0$  to find the constants in  $\phi$ .
- Write the expression for the stresses.
- Write the expression for the strains.
- Find the expressions for the displacements.

## SOLUTION

### Boundary Conditions

a) The boundary conditions are on follows

Top

$$\sigma_{yy}(x, -h) = 0, 0 < x < L \quad (1)$$

$$\tau_{xy}(x, -h) = 0, 0 < x < L \quad (2)$$

Bottom

$$\sigma_{yy}(x, h) = 0, 0 < x < L \quad (3)$$

$$\tau_{xy}(x, h) = 0, 0 < x < L \quad (4)$$

Left

$$P_x : \int_{-h}^h \sigma_{xx}(0, y) t \, dy = 0 \quad (5)$$

$$P_y : \int_{-h}^h \tau_{xy}(0, y) t \, dy = -P \quad (6)$$

$$M : \int_{-h}^h \sigma_{xx}(0, y) t y \, dy = 0 \quad (7)$$

Right

$$P_x : \int_{-h}^h \sigma_{xx}(L, y) t \, dy = 0 \quad (8)$$

$$P_y : \int_{-h}^h \tau_{xy}(L, y) t \, dy = -P \quad (9)$$

$$M : \int_{-h}^h \sigma_{xx}(L, y) y t \, dy = -PL \quad (10)$$

**Expressions for stresses**

$$\begin{aligned} \text{b) } \sigma_x &= \frac{\partial^2 \phi}{\partial y^2} \\ &= 2d_3x^2 + 6d_4xy + 12d_5y^2 + 2c_3x + 6c_4y + 2b_3 \end{aligned} \quad (11)$$

$$\begin{aligned} \sigma_y &= \frac{\partial^2 \phi}{\partial x^2} \\ &= 12d_1x^2 + 6d_2xy + 2d_3y^2 + 6c_1x + 2c_2y + 2b_1 \end{aligned} \quad (12)$$

$$\begin{aligned} \tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} \\ &= -(3d_2x^2y + 4d_3xy + 3d_4y^2 + 2c_2x + 2c_3y + b_2) \end{aligned} \quad (13)$$

### Applying Boundary Conditions To Find The Constants

c) Apply (1) and (3), that is,

$$\sigma_{yy}(x, -h) = 0, 0 < x < L \quad (1)$$

$$\sigma_{yy}(x, h) = 0, 0 < x < L \quad (2)$$

gives

$$12d_1x^2 - 6d_2xh + 2d_3h^2 + 6c_1x - 2c_2h + 2b_1 = 0$$

$$12d_1x^2 + 6d_2xh + 2d_3h^2 + 6c_1x + 2c_2h + 2b_1 = 0$$

The above equations by collecting terms in x, give,

$$d_1 = 0, 6d_2h + 6c_1 = 0, 2d_3h^2 + 2c_2h + 2b_1 = 0,$$

$$-6d_2h + 6c_1 = 0, 2d_3h^2 - 2c_2h + 2b_1 = 0.$$

which further give

$$d_1 = 0, d_2 = 0, c_1 = 0, c_2 = 0, b_1 = -d_3h^2 \quad (14)$$

Apply boundary condition (2) and (4), that is

$$\tau_{xy}(x, -h) = 0, 0 < x < L \quad (2)$$

$$\tau_{xy}(x, h) = 0, 0 < x < L \quad (4)$$

gives

$$-(-3d_2x^2h - 4d_3xh + 3d_4h^2 + 2c_2x - 2c_3h + b_2) = 0$$

$$-(3d_2x^2h + 4d_3xh + 3d_4h^2 + 2c_2x + 2c_3h + b_2) = 0$$

The above equations by collecting terms in x give

$$d_2 = 0, 4d_3h - 2c_2 = 0, 3d_4h^2 - 2c_3h + b_2 = 0$$

$$-4d_3h - 2c_2 = 0, 3d_4h^2 + 2c_3h + b_2 = 0$$

which further give

$$d_2 = 0, d_3 = 0, c_2 = 0, c_3 = 0, b_2 = -3d_4h^2 \quad (15)$$

Since  $d_3 = 0$  from equation (15), from equation (14),

$$b_1 = 0. \quad (16)$$

Substituting these to simplify expression for stresses gives

$$\begin{aligned}\sigma_{xx} &= 6d_4xy + 12d_5y^2 + 6c_4y + 2b_3 \\ \sigma_{yy} &= 0 \\ \tau_{xy} &= -3d_4(y^2 - h^2)\end{aligned}$$

Before applying the boundary conditions on the left and right edges, let us check if

$$\nabla^4\phi = 0$$

$$24d_1 + 4d_3 + 24d_5 = 0$$

Since  $d_1 = 0$  from equation (14), and  $d_3 = 0$  from equation (15), then from above

equation

$$d_5 = 0 \tag{17}$$

This further simplifies the expressions for the stresses as

$$\begin{aligned}\sigma_{xx} &= 6d_4xy + 6c_4y + 2b_3 \\ \sigma_{yy} &= 0 \\ \tau_{xy} &= -3d_4(y^2 - h^2)\end{aligned}$$

Apply boundary condition (5), that is

$$\int_{-h}^h \sigma_{xx}(0, y)tdy = 0$$

$$\int_{-h}^h (6c_4y + 2b_3)tdy = 0$$

$$4b_3ht = 0$$

$$b_3 = 0 \tag{18}$$

Apply boundary condition (7), that is

$$\begin{aligned}
 \int_{-h}^h \sigma_{xx}(0, y)tydy &= 0 \\
 \int_{-h}^h (6c_4y + 2b_3)tydy &= 0 \\
 4c_4h^3t &= 0 \\
 c_4 &= 0
 \end{aligned} \tag{19}$$

Apply boundary condition (6), that is

$$\begin{aligned}
 \int_{-h}^h \tau_{xy}(0, y)tdy &= -P \\
 \int_{-h}^h -3d_4(y^2 - h^2)tdy &= -P \\
 -3d_4t \left[ \frac{2h^3}{3} - 2h^3 \right] &= -P \\
 d_4 &= -\frac{P}{4h^3t} \\
 &= -\frac{P}{6I}
 \end{aligned} \tag{20}$$

as the second moment of area

$$I = \frac{1}{12}t(2h)^3 = \frac{2}{3}th^3$$

### Writing The Stress Expressions

d) So the stresses are found now with equation (18), (19) and (20) as

$$\begin{aligned}\sigma_{xx} &= -\frac{P}{I}xy \\ \sigma_{yy} &= 0 \\ \tau_{xy} &= \frac{P}{2I}(y^2 - h^2)\end{aligned}$$

So the only conditions left to check are the boundary conditions on the right, even when we have found all the constants of stresses.

Check boundary conditions (8), that is

$$\begin{aligned}\int_{-h}^h \sigma_{xx}(L, y)tdy \\ = \int_{-h}^h -\frac{P}{I}Ly tdy \\ = 0 \text{ satisfied}\end{aligned}$$

Apply boundary condition (9),

$$\begin{aligned}\int_{-h}^h \tau_{xy}(L, y)tdy \\ = \int_{-h}^h \frac{P}{2I}(y^2 - h^2)tdy \\ = \frac{P}{2I}\left(\frac{2h^3}{3} - 2h^3\right)t \\ = \frac{P}{2I}\left(\frac{-4h^3t}{3}\right) \quad \left(I = \frac{2}{3}th^3\right) \\ = -P \text{ (satisfied)}\end{aligned}$$

Apply boundary condition (10),

$$\begin{aligned}\int_{-h}^h \sigma_{xx}(L, y)ytdy \\ = \int_{-h}^h -\frac{P}{I}Ly^2t dy \\ = -\frac{PLt}{I} \frac{2h^3}{3} \quad \left(I = \frac{2}{3}th^3\right) \\ = -PL \text{ (satisfied)}\end{aligned}$$

### Expression for strains

e) The strains are found as

From eqns (21), (22) and (23), using Hooke's law

$$\begin{aligned}\epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) \\ &= \frac{1}{E}\left(-\frac{Pxy}{I} - \nu 0\right) \\ \epsilon_x &= -\frac{Pxy}{EI}\end{aligned}\tag{24}$$

$$\begin{aligned}\epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) \\ &= \frac{1}{E}\left[0 - \nu\left(-\frac{Pxy}{I}\right)\right] \\ \epsilon_y &= \frac{\nu Pxy}{EI}\end{aligned}\tag{25}$$

$$\begin{aligned}\gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \gamma_{xy} &= \frac{P}{2IG}(y^2 - h^2)\end{aligned}\tag{26}$$

f) Displacements

$$\frac{\partial u}{\partial x} = \epsilon_x = -\frac{Pxy}{EI} \quad (27)$$

$$\frac{\partial v}{\partial x} = \epsilon_y = \frac{vPxy}{EI} \quad (28)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy} = \frac{P}{2IG}(y^2 - h^2) \quad (29)$$

From eqn (27)

$$u = -\frac{Px^2y}{2EI} + g_1(y)$$

From eqn (28)

$$v = \frac{vPxy^2}{2EI} + g_2(x)$$

Substituting above two expressions for u and v in eqn(29)

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy} = \frac{P}{2IG}(y^2 - h^2) = \frac{P(1+v)}{EI}(y^2 - h^2)$$

$$\text{as } G = \frac{E}{2(1+v)}$$

$$-\frac{Px^2}{2EI} + \frac{dg_1(y)}{dy} + \frac{vPy^2}{2EI} + \frac{dg_2(x)}{dx} = \frac{P(1+v)}{EI}(y^2 - h^2)$$

Below left hand side is only a function of 'y' and right hand side is a function of 'x'

$$\frac{dg_1(y)}{dy} + \frac{vPy^2}{2EI} - \frac{P(1+v)}{EI}(y^2 - h^2) = \frac{Px^2}{2EI} - \frac{dg_2(x)}{dx} = a_1$$

$$\begin{aligned} \frac{dg_1(y)}{dy} &= -\frac{vPy^2}{2EI} + \frac{P(1+v)}{EI}(y^2 - h^2) + a_1 \\ &= \frac{P}{EI} \left(1 + \frac{v}{2}\right) y^2 - \frac{P(1+v)}{EI} h^2 + a_1 \end{aligned}$$

$$g_1(y) = \frac{P}{EI} \left(1 + \frac{v}{2}\right) \frac{y^3}{3} - \frac{P(1+v)}{EI} h^2 y + a_1 y + a_2$$

$$\frac{dg_2(x)}{dx} = \frac{Px^2}{2EI} - a_1$$

$$g_2(x) = \frac{Px^3}{6EI} - a_1 x + a_3$$

$$u = \frac{Px^2y}{2EI} + \frac{P}{EI} \left(1 + \frac{v}{2}\right) \frac{y^3}{3} - \frac{P(1+v)}{EI} h^2 y + a_1 y + a_2$$

$$v = \frac{vPxy^2}{2EI} + \frac{Px^3}{6EI} - a_1 x + a_3$$

**Apply the boundary conditions on the displacements**

$$u(L,0) = 0$$

$$v(L,0) = 0$$

$$\frac{\partial v}{\partial x}(L,0) = 0$$

$$u(L,0) = 0 \text{ gives } a_2 = 0$$

$$v(L,0) = 0 \text{ gives } \frac{P}{6EI}L^3 - a_1L + a_3 = 0$$

$$\frac{\partial v}{\partial x} = \frac{\nu Py^2}{2EI} + \frac{Px^2}{2EI} - a_1$$

$$\frac{\partial v}{\partial x}(L,0) = \frac{PL^2}{2EI} - a_1 = 0$$

$$a_1 = \frac{PL^2}{2EI}$$

$$\frac{PL^3}{6EI} - \frac{PL^2}{2EI}L + a_3 = 0$$

$$a_3 = \frac{PL^3}{3EI}$$

$$u = -\frac{Px^2y}{2EI} + \frac{Py^3}{3EI} \left(1 + \frac{\nu}{2}\right) - \frac{P(1+\nu)h^2y}{EI} + \frac{PL^2y}{2EI}$$

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} - \frac{PL^2x}{2EI} + \frac{PL^3}{3EI}$$

**Mechanics of Materials Solution**

$$EI \frac{d^2v}{dx^2} = M = Px$$

$$\frac{dv}{dx} = \frac{Px^2}{2EI} + c_1$$

$$v = \frac{Px^3}{6EI} + c_1x + c_2$$

$$v(x=L) = 0$$

$$\frac{dv}{dx}(x=L) = 0$$

$$\frac{PL^3}{6EI} + c_1L + c_2 = 0$$

$$\frac{PL^2}{2EI} + c_1 = 0$$

$$c_1 = -\frac{PL^2}{2EI}$$

$$c_2 = \frac{PL^3}{3EI}$$

$$v = \frac{Px^3}{6EI} - \frac{PL^2x}{2EI} + \frac{PL^3}{3EI}$$