Question

At a point in a loaded member, the stresses relative to an *x,y,z* coordinate system are given by $\begin{bmatrix} 60 & 20 & 10 \\ 20 & -40 & -5 \\ 10 & -5 & 30 \end{bmatrix}$ MPa. Calculate the magnitude and direction of maximum principal stress.

Solution

σ_x = 60MPa, σ_y = -40 MPa, σ_z =30 MPa; τ_{xy}= 20 MPa, τ_{xz} =10 MPa, τ_{yz}= -5 MPa;

Let us denote the normal stresses in the *x, y, z* direction as σ_x, σ_y, σ_z respectively and let the corresponding shear stresses be τ_{xy}, τ_{zy}, τ_{xz}, respectively.

We first find the *stress invariants*, I_1, I_2, and I_3

$I_1 = \sigma_x + \sigma_y + \sigma_z$

= 60+(-40)+30

=50

$I_2 = (\sigma_x \times \sigma_y) + (\sigma_y \times \sigma_z) + (\sigma_y \times \sigma_z) - (\tau^2_{xy} + \tau^2_{xz} + \tau^2_{yz})$

$= 60 \times (-40) + (-40) \times 30 + (-40) \times (30) - (20^2 + 10^2 + (-5)^2)$

$= -2325 \text{ MPa}^2$

$$I_3 = \begin{vmatrix} 60 & 20 & 10 \\ 20 & -40 & -5 \\ 10 & -5 & 30 \end{vmatrix}$$

$= -83500 \text{ MPa}^3$

Now, we write the Equation (1.28) from the text which when solved would give us three principal stresses

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$$

$$\therefore \sigma_p^3 - 50\sigma_p^2 - 2325\sigma_p + 83500 = 0$$

Now, to solve this cubic equation, we use Newton-Raphson's Method, we assume an initial value for σ_p and then using

$$\sigma_p^{'} = \sigma_p - \frac{f(\sigma_p)}{f^{'}(\sigma_p)}$$

Here,

$$f(\sigma_p) = \sigma_p^3 - 50\sigma_p^2 - 2325\sigma_p + 83500$$

And

$$f^{'}(\sigma_p) = 3\sigma_p^2 - 100\sigma_p - 2325$$

Let us start with an initial value of 60,

$$\therefore \sigma_p' = 60 - \frac{-20000}{2475}$$

$$= 68.0808$$

Now, we will repeat the above exercise with a value of 68.0808.This is continued up till the absolute relative percentage error, is lesser than a fixed allowable tolerance value say 0.005%, where the absolute relative percentage error, $|\varepsilon_a|$

$$\varepsilon_a = \left| \frac{\sigma_{p|new} - \sigma_{p|old}}{\sigma_{p|new}} \right| \times 100$$

Ultimately, this gives us a value of 66.0611 MPa

Now, we do polynomial division in order to get a quadratic equation, which will be solved later to give the other two factors.

$$\therefore \frac{\sigma_p^3 - 50\sigma_p^2 - 2325\sigma_p + 83500}{\sigma_p - 66.0611}$$

$$= \sigma_p^2 + 16.06\sigma_p - 1264$$

Now, we solve this quadratic equation $\sigma_p^2 + 16.06\sigma_p - 1264 = 0$ to get

$$\sigma_p = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\sigma_p = \frac{-(16.06) \pm \sqrt{(16.06)^2 - 4 \times 1 \times (-1264)}}{2 \times 1}$$

$$\sigma_p = \frac{-16.06 \pm 72.896}{2 \times 1}$$

$$\therefore \sigma_p = 28.41\,\text{MPa},\ -44.478\,\text{MPa}$$

We note that the maximum of the three principal stresses is $\sigma_p = 66.061\,\text{MPa}$

Now, in order to find its direction, we need to find its direction cosines, *l*, *m* and *n*.

We expand the first two rows of the matrix

$$\begin{bmatrix} \sigma_x - \sigma_p & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_p & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_p \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 60 - 66.061 & 20 & 10 \\ 20 & -40 - 66.061 & -5 \\ 10 & -5 & 30 - 66.061 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the first two rows, we get:

$$-6.061l + 20.m + 10.n = 0 \quad (1)$$

$$20l - 106.06m - 5n = 0 \quad (2)$$

And the third equation is obtained from the geometric condition that

$$l^2 + m^2 + n^2 = 1 \quad (3)$$

This is done so as to avoid getting a zero solution for *l, m* and *n*.

First, we eliminate *n* from (1) and (2)

$$33.939l - 192.12m = 0 \quad (4)$$

Now, we eliminate m from (1) and (2)

$$-247.72l + 960.06n = 0 \quad (5)$$

Now, substitute values of *m* and *n* in terms of *l* from (4) and (5) into (3) to get:

$$l^2 + (0.1766l)^2 + (0.258l)^2 = 1$$

which gives

$$l = 0.954$$

put this back in equations (4) and (5), to get:

$$m = 0.168$$

$$n = 0.246$$

Maximum principal stress is 66.061 MPa, and its three direction cosines *l, m* and *n* are 0.954, 0.168 and 0.246, respectively.