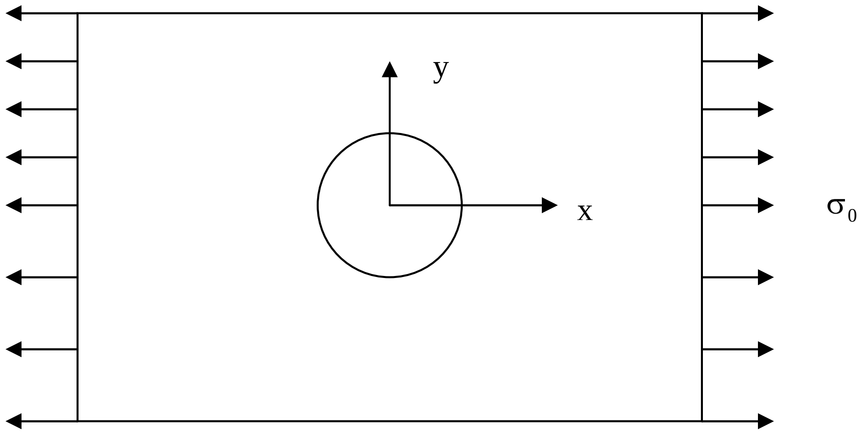
Problem: Find the stress state in a plate with a circular hole of radius a. The plate is subjected to a simple tensile stress of σ_0.



Solution:

The boundary conditions for the problem are

On the edge of the hole at r = a,

$$\begin{aligned} \sigma_{rr}(a,\theta) &= 0, \qquad 0 \le \theta \le 2\pi \\ \tau_{r\theta}(a,\theta) &= 0, \qquad 0 \le \theta \le 2\pi \end{aligned} \qquad \text{(a)}$$

and on the edges of the infinite plate, that is $r \to \infty$ $(x \to \infty, y \to \infty)$

At the right and left edges

$$\sigma_{xx}(x,y) = \sigma_0, x \to \pm\infty$$

$$\tau_{xy}(x,y) = 0, x \to \pm\infty$$

At the top and bottom edges

$$\sigma_{yy}(x,y) = 0, y \to \pm\infty \qquad \text{(b)}$$

$$\tau_{xy}(x,y) = 0, y \to \pm\infty$$

The solution to the problem of a plate with no hole is given by the stress function

$$\begin{aligned} \phi &= \frac{1}{2}\sigma_0 y^2 \\ \phi &= \frac{1}{4}\sigma_o r^2(1-\cos 2\theta) \end{aligned} \qquad (1)$$

Hence the stresses in the problem of a plate with no hole subjected to a uniaxial tensile stress is

$$\sigma_r = \frac{1}{2}\sigma_0(1+\cos 2\theta)$$
$$\sigma_\theta = \frac{1}{2}\sigma_0(1-\cos 2\theta), \qquad (2)$$
$$\tau_{r\theta} = -\frac{1}{2}\sigma_0 \sin 2\theta$$

The above will be the stress state far from the center of the hole. Hence, the stress function can be assumed to be of the form

$$\phi = f_1(r) + f_2(r)\ \text{Cos}2\theta$$

Hence $\nabla^4\phi = 0$ gives

$$\nabla^4\left[f_1(r) + f_2(r)\ \text{Cos}2\theta\right] = 0 \qquad (3)$$

$$\nabla^4 f_1(r) + \nabla^4\left[f_2(r)\ \text{Cos}2\theta\right] = 0$$

Since the first term does not include 'θ' terms and the second term does,

$$\nabla^4 f_1(r) = 0$$
$$\nabla^4\left[f_2(r)\text{Cos}2\theta\right] = 0 \qquad (4)$$

respectively, give

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 f_1}{dr^2} + \frac{1}{r}\frac{df_1}{dr}\right) = 0$$
$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2 f_2}{dr^2} + \frac{1}{r}\frac{df_2}{dr} - \frac{4f_2}{r^2}\right) = 0 \qquad (5)$$

The above two linear ordinary differential equations have the solution respectively as

$$f_1 = c_1 r^2 \ln r + c_2 r^2 + c_3 \ln r + c_4$$

$$f_2 = c_5 r^2 + c_6 r^4 + \frac{c_7}{r^2} + c_8$$

Using $\phi = f_1(r) + f_2(r)\ \text{Cos}2\theta$ and the stress-Airy Stress Function relationships

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$
$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \qquad (6)$$

$$\tau_r = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) \tag{7}$$

we get

$$\sigma_r = c_1(1 + 2\ln r) + 2c_2 + \frac{c_3}{r^2} - \left(2c_5 + \frac{6c_7}{r^4} + \frac{4c_8}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = c_1(3 + 2\ln r) + 2c_2 - \frac{c_3}{r^2} + \left(2c_5 + 12c_6 r^2 + \frac{6c_7}{r^4}\right)\cos 2\theta \tag{8}$$

$$\tau_{r\theta} = \left(2c_5 + 6c_6 r^2 - \frac{6c_7}{r^4} - \frac{2c_8}{r^2}\right)\sin 2\theta$$

As $r \to \infty$, that is the far away from the hole, the stresses are

$$\sigma_r = c_1(1 + 2\ln r) + 2c_2 - 2c_5\ Cos2\theta$$
$$\sigma_\theta = c_1(3 + 2\ln r) + 2c_2 + \left(2c_5 + 12c_6 r^2\right)\cos 2\theta \tag{9}$$
$$\tau_{r\theta} = (2c_5 + 6c_6 r^2)\ Sin2\theta$$

Looking at the far field stress, that is the stresses far way from the center of hole given by Equation (2) gives c_1 =0 and c_6 =0 (since as $r \to \infty, \ln r \to \infty, r^2 \to \infty)$.

From the boundary conditions (Equation (a)) at r = a give

$$2c_2 + \frac{c_3}{a^2} = 0, \qquad 2c_5 + \frac{6c_7}{a^4} + \frac{4c_8}{a^2} = 0, \qquad 2c_5 - \frac{6c_7}{a^4} - \frac{2c_8}{a^2} = 0$$

From the stress state far away from the center (Equation (2))

$$2c_2 = \frac{\sigma_0}{2}, -2c_5, = \frac{\sigma_0}{2}$$

The constants are hence given by

$$c_1 = 0$$

$$c_2 = \frac{\sigma_0}{4}$$

$$c_3 = -2c_2 a^2 = -\frac{\sigma_0}{2}a^2$$

$$c_5 = -\frac{\sigma_0}{4}$$

$$c_6 = 0$$

$$c_7 = \frac{a^4}{2}$$

$$c_8 = -a^2$$

From Equation (8), the stress state then is

$$\sigma_r = \frac{1}{2}\sigma_o\left[\left(1 - \frac{a^2}{r^2}\right) + \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta\right]$$

$$\sigma_\theta = \frac{1}{2}\sigma_o\left[\left(1 + \frac{a^2}{r^2}\right) - \left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta\right]$$

$$\tau_{r\theta} = -\frac{1}{2}\sigma_o\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

$$(\sigma_\theta)_{max} = 3\sigma_o, \qquad \theta = \pm\pi/2$$

$$(\sigma_\theta)_{min} = -\sigma_o, \qquad \theta = 0,\ \theta = \pm\pi$$

Appendix A

Question:
How does $\nabla^4 f_1(r) = 0$ give the solution as $f_1(r) = c_1 r^2 \ln r + c_2 r^2 + c_3 \ln r + c_4$?

Answer:

$$\nabla^4 f_1(r) = 0 \tag{A-1}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) f_1(r) = 0$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f_1}{\partial \theta^2}\right) = 0 \tag{A-2}$$

Since $f_1(r)$ is a function of 'r' only,

$$\frac{\partial^2 f_1}{\partial \theta^2} = 0$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r}\right) = 0$$

$$\frac{\partial^4 f_1}{\partial r^4} + \frac{1}{r}\frac{\partial^3 f_1}{\partial r^3} + \frac{1}{r^2}\frac{\partial^4 f_1}{\partial \theta^2 \partial r^2} + \frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial f_1}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{1}{r}\frac{\partial f_1}{\partial r}\right) = 0 \tag{A-3}$$

Since f_1 is a function of 'r' only,

$$\frac{\partial^4 f_1}{\partial \theta^2 \partial r^2} = 0,$$

and $\frac{\partial^2}{\partial \theta^2}\left(\frac{1}{r}\frac{\partial f_1}{\partial r}\right) = 0,$

gives

$$\frac{\partial^4 f_1}{\partial r^4} + \frac{1}{r}\frac{\partial^3 f_1}{\partial r^3} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 f_1}{\partial r^2} - \frac{1}{r^2}\frac{\partial f_1}{\partial r}\right) + \frac{1}{r}\left(\frac{1}{r}\frac{\partial f_1}{\partial r}\right) = 0$$

$$\frac{\partial^4 \mathrm{f}_1}{\partial \mathrm{r}^4} + \frac{1}{\mathrm{r}}\frac{\partial^3 \mathrm{f}_1}{\partial \mathrm{r}^3} + \left(-\frac{1}{\mathrm{r}^2}\frac{\partial^2 \mathrm{f}_1}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial^3 \mathrm{f}_1}{\partial \mathrm{r}^3} + \frac{2}{\mathrm{r}^3}\frac{\partial \mathrm{f}_1}{\partial \mathrm{r}} - \frac{1}{\mathrm{r}^2}\frac{\partial^2 \mathrm{f}_1}{\partial \mathrm{r}^2} \right) + \frac{1}{\mathrm{r}^2}\frac{\partial^2 \mathrm{f}_1}{\partial \mathrm{r}^2} - \frac{1}{\mathrm{r}^3}\frac{\partial \mathrm{f}_1}{\partial \mathrm{r}} = 0$$

$$\frac{\partial^4 f_1}{\partial r^4} + \frac{2}{r}\frac{\partial^3 f_1}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r^3}\frac{\partial f_1}{\partial r} = 0 \qquad \text{(A-4)}$$

$$\frac{d^4 f_1}{dr^4} + \frac{2}{r}\frac{d^3 f_1}{dr^3} - \frac{1}{r^2}\frac{d^2 f_1}{dr^2} + \frac{1}{r^3}\frac{df_1}{dr} = 0$$

Assume $r = e^t$ (A-5)

Then $t = ln\ r$ (A-6)

$$\frac{df_1}{dr} = \frac{df_1}{dt}\frac{dt}{dr} = \frac{1}{r}\frac{df_1}{dt}$$

$$\frac{d^2 f_1}{dr^2} = \frac{d}{dr}\left(\frac{df_1}{dr}\right) = \frac{d}{dr}\left(\frac{1}{r}\frac{df_1}{dt}\right) = \frac{1}{r^2}\left(\frac{d^2 f_1}{dr^2} - \frac{df_1}{dt}\right)$$

Similarly

$$\frac{d^3 f_1}{dr^3} = \frac{1}{r^3}\left(\frac{d^3 f_1}{dt^3} - 3\frac{d^2 f_1}{dt^2} + 2\frac{df_1}{dt}\right)$$

$$\frac{d^4 f_1}{dr^4} = \frac{1}{r^4}\left(\frac{d^4 f_1}{dt^4} - 6\frac{d^3 f_1}{dt^3} + 11\frac{d^2 f_1}{dt^2} - 6\frac{df_1}{dt}\right)$$

Substituting these in the expression (A-4) gives

$$\frac{1}{r^4}\left(\frac{d^4 f_1}{dt^4} - 6\frac{d^3 f_1}{dt^3} + 11\frac{d^2 f_1}{dt^2} - 6\frac{df_1}{dt}\right) + \frac{2}{r}\frac{1}{r^3}\left(\frac{d^3 f_1}{dt^3} - 3\frac{d^2 f_1}{dt^2} + 2\frac{df_1}{dt}\right) - \frac{1}{r^2}\frac{1}{r^2}\left(\frac{d^2 f_1}{dr^2} - \frac{df_1}{dt}\right) + \frac{1}{r^3}\frac{1}{r}\frac{df_1}{dt} = 0$$

$$\frac{d^4 f_1}{dt^4} - 4\frac{d^3 f_1}{dt^3} + 4\frac{d^2 f_1}{dt^2} = 0 \qquad \text{(A-7)}$$

This is a linear ordinary differential equation but with fixed constants. The characteristic equation then is:

$$m^4 - 4m^3 + 4m^2 = 0$$
$$m^2\left(m^2 - 4m + 4\right) = 0$$
$$m^2(m-2)(m-2) = 0$$

So

m = 0, 0, 2, 2 are the roots of the characteristic equation.

$$f_1 = c_1 t e^{2t} + c_2 e^{2t} + c_3 t + c_4$$
$$f_1(r) = c_1(\ln r)r^2 + c_2 r^2 + c_3 \ln r + c_4 \qquad \text{(A-8)}$$
$$f_1(r) = c_1 r^2 \ln r + c_2 r^2 + c_3 \ln r + c_4$$

Here is another way to solve equation (A-4)

$$\frac{d^4 f_1}{dr^4} + \frac{2}{r}\frac{d^3 f_1}{dr^3} - \frac{1}{r^2}\frac{d^2 f_1}{dr^2} + \frac{2}{r^3}\frac{df_1}{dr} = 0$$

The above expression is same as

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{d^2 f_1}{dr^2} + \frac{1}{r}\frac{df_1}{dr}\right) = 0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\left(\frac{d^2 f_1}{dr^2} + \frac{1}{r}\frac{df_1}{dr}\right) = 0$$

$$\frac{d}{dr}\left(r\frac{d}{dr}\right)\left(\frac{d^2 f_1}{dr^2} + \frac{1}{r}\frac{df_1}{dr}\right) = 0$$

Integrating both sides

$$\left(\mathrm{r}\frac{\mathrm{d}}{\mathrm{dr}}\right)\left(\frac{\mathrm{d}^2\mathrm{f}_1}{\mathrm{dr}^2} + \frac{1}{\mathrm{r}}\frac{\mathrm{df}_1}{\mathrm{dr}}\right) = \mathrm{a}_1$$

$$\frac{\mathrm{d}}{\mathrm{dr}}\left(\frac{\mathrm{d}^2\mathrm{f}_1}{\mathrm{dr}^2} + \frac{1}{\mathrm{r}}\frac{\mathrm{df}_1}{\mathrm{dr}}\right) = \frac{\mathrm{a}_1}{\mathrm{r}}$$

Integrating both sides

$$\frac{d^2 f_1}{dr^2} + \frac{1}{r}\frac{df_1}{dr} = a_1 \ln r + a_2$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_1}{dr}\right) = a_1 \ln r + a_2$$

$$\frac{d}{dr}\left(r\frac{df_1}{dr}\right) = a_1 r \ln r + a_2 r$$

Integrating both sides

$$r\frac{df_1}{dr} = a_1\left(\frac{r^2}{2}\ln r - \frac{1}{4}\right) + a_2\frac{r^2}{2} + a_3$$

$$\frac{df_1}{dr} = a_1\left(\frac{r}{2}\ln r - \frac{1}{4r}\right) + a_2\frac{r}{2} + \frac{a_3}{r}$$

$$f_1 = \frac{a_1}{2}\left(\frac{r^2}{2}\ln r - \frac{1}{4}\right) - \frac{a_1}{4}\ln r + a_2\frac{r^2}{4} + a_3 \ln r$$

Integrating both sides

$$f_1 = \frac{a_1}{4}\left(r^2 \ln r\right) + \left(-\frac{a_1}{4} + a_3\right)\ln r + a_2\frac{r^2}{4} - \frac{a_1}{8}$$

$$f_1 = c_1 r^2 \ln r + c_2 \ln r + c_3 r^2 + c_4 \qquad \text{(A-9)}$$

The above expression is same as obtained in expression (A-8).

<u>Exercise:</u> How does $\nabla^4\left[f_2(r)\,\text{Cos}2\theta\right] = 0$ give the solution $f_2(r) = c_5 r^2 + c_6 r^2 + \frac{c_7}{r^2} + c_8$. *Hint:* Follow the same procedure as given for $\nabla^4 f_1(r) = 0$ and assume $f_2(r) = \sum_{n=-\infty}^{n=\infty} A_n r^n$ as the solution. What do you get for values of A_n?

<u>Exercise:</u> Why could I not chose $f_1(r) = \sum_{n=-\infty}^{\infty} A_n r^n$ for solving $\nabla^4 f_1(r) = 0$?