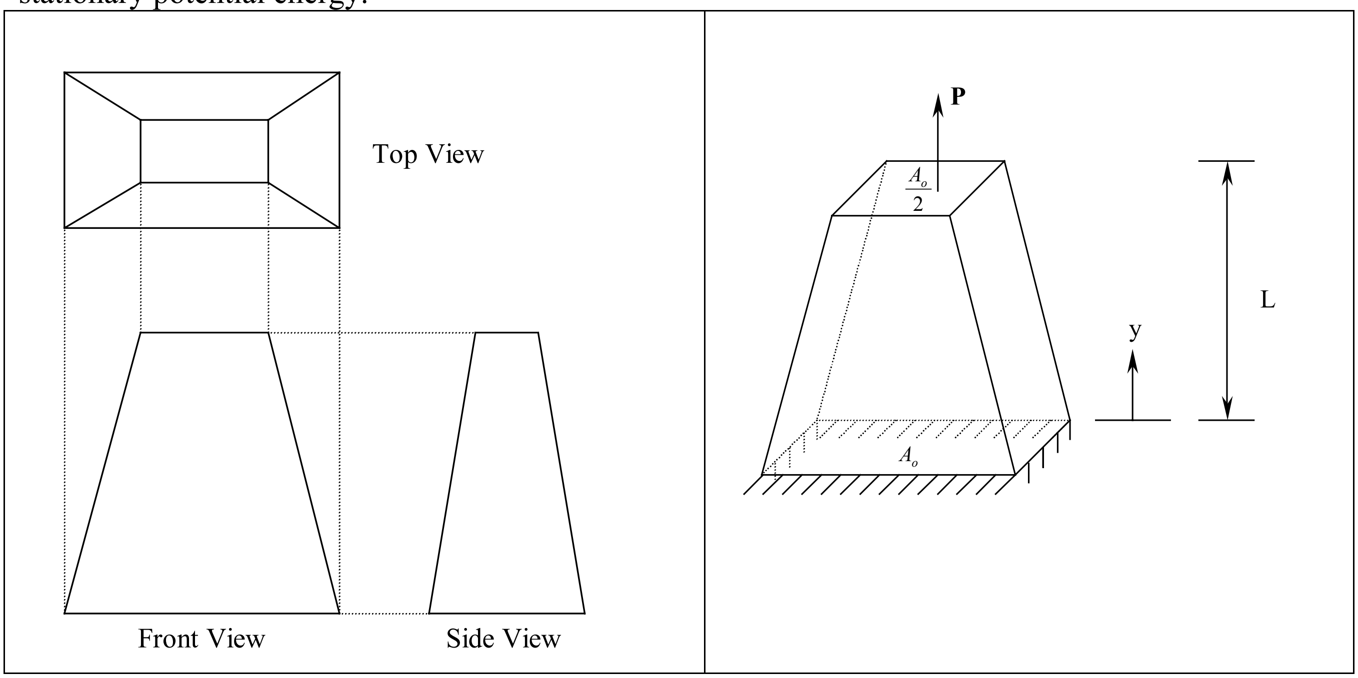
Principle of Stationery Potential Energy

Among all admissible displaced configurations of a conservative system those that satisfy equations of equilibrium make the potential energy stationery with respect to small admissible variations of displacement. If the stationery condition is a relative minimum, the equilibrium state is stable.

Example:

A tapered bar as shown in the figure is subject to an axial load 'P'. The area of the tapered bar varies linearly from y = 0 to y = L, from, A_o to $\frac{A_o}{2}$. Assuming an admissible displacement of $v(y) = b_0 y + c_o y^2$, find the displacement of y = L using the principle of stationary potential energy.



Assume an expression for the area 'A' as

$$A = my + c$$

then

$$A_o = m(0) + c$$

$$\frac{A_o}{2} = m(L) + c$$

$$c = A_0$$

$$m = -\frac{A_o}{2L}$$

$$A = -\frac{A_o}{2L}y + A_o$$

$$= \frac{A_o}{2L}(2L - y)$$

$$U = \int_V \frac{1}{2} \sigma_y \in_y \, dV$$

$$= \int_V \frac{1}{2} \mathrm{E} \in_y \in_y \, dV$$

$$= \int_V \frac{1}{2} \mathrm{E} \left(\frac{dv}{dy} \right) \left(\frac{dv}{dy} \right) dV$$

$$= \int_V \frac{1}{2} \mathrm{E} \left(\frac{dv}{dy} \right)^2 dV$$

$$= \int_{o \cdot V}^{V} \frac{1}{2} \mathrm{E} \left(\frac{dv}{dy} \right)^2 A dy$$

$$W = P v_L$$

$$\Pi = U - W$$

$$= \int_0^L \frac{1}{2} E \left(\frac{dv}{dy} \right)^2 A dy - P v_L$$

Assuming

$$v = b_0 y + c_o y^2$$

$$\frac{dv}{dy} = b_o + 2 c_o y$$

$$= \left[\int_0^L \frac{1}{2} \mathrm{E} (b_o + 2 c_o y)^2 \left(\frac{A_o}{2L} (2L - y) \right) dy \right] - P \left(b_o L + c_o L^2 \right)$$

$$= \frac{E A_o L}{24} \left(10 L^2 c_o{}^2 + 16 b_o c_o L + 9 b_o{}^2 \right) - P \left(b_o L + c_o L^2 \right)$$

$$\frac{\partial \Pi}{\partial b_o} = 0$$

gives

$$8 \mathrm{E} A_o c_o L + 9 \mathrm{E} A_o b_o - 12 P = o \text{..........(1)}$$

$$\frac{\partial \Pi}{\partial c_o} = 0$$

gives

$$5 \mathrm{E} A_o c_o L + 4 \mathrm{E} A_o b_o - 6 P = 0 \text{..........(2)}$$

From (1) and (2), we get

$$b_o = \frac{12}{13}\frac{P}{\mathrm{E}A_o}$$

$$c_o = \frac{6}{13}\frac{P}{EA_o L}$$

Hence

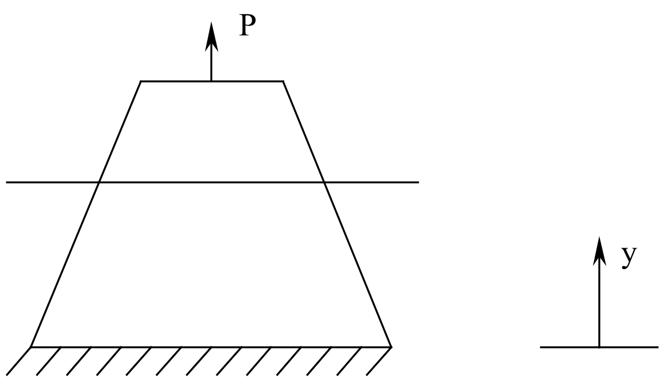
$$v = \frac{12}{13}\frac{P}{\mathrm{E}A_o} y + \frac{6}{13}\frac{P}{EA_o L} y^2$$

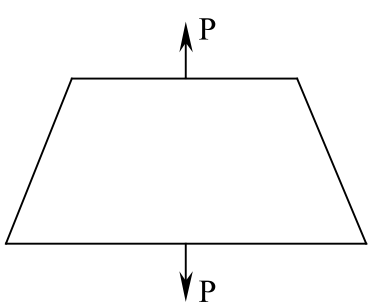
$$v_L = \frac{12}{13}\frac{P}{\mathrm{E}A_o} L + \frac{6}{13}\frac{P}{EA_o L} L^2$$

$$= \frac{18}{13}\frac{PL}{EA_o}$$

$$= 1.38462\frac{PL}{EA_o}$$

Exact Solution:





$$\in_y = \frac{\sigma_y}{\mathrm{E}}$$

$$= \frac{P}{A\mathrm{E}}$$

$$= \frac{P}{\frac{A_o}{2L}(2L - x)\mathrm{E}}$$

$$\frac{dv}{dy} = \frac{2PL}{A_o(2L - x)\text{E}}$$

$$v_y - v_{y=0} = \int_0^y \frac{2PL}{A_o(2L - y')\text{E}} dy'$$

$$= \frac{2PL}{A_o\text{E}} \ln\left(\frac{2L}{2L - y}\right)$$

$$v_y = \frac{2PL}{A_o\text{E}} \ln\left(\frac{2L}{2L - y}\right)$$

$$v_L = \frac{PL}{A\text{E}}(2\ln 2)$$

$$= 1.38629 \frac{PL}{\text{E}A_o}$$

The absolute relative true error is

$$|\in_t| = \left|\frac{Exact\ Value - Aprox\ Value}{Exact\ value}\right| \times 100$$

$$|\in_t| = \left|\frac{\frac{PL}{A_o\text{E}}(2\ln 2) - \frac{18}{13}\frac{PL}{A_o\text{E}}}{\frac{PL}{A_o\text{E}}(2\ln 2)}\right| \times 100$$

$$= 0.11\%$$