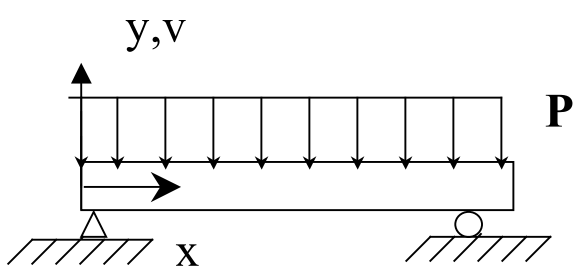
Principle of Stationery Potential Energy

Among all admissible displaced configurations of a conservative system those that satisfy equations of equilibrium make the potential energy stationery with respect to small admissible variations of displacement. If the stationery condition is a relative minimum, the equilibrium state is stable.

Example:

For the beam shown, using an admissible displacement function $v(x) = a_1 x(L - x)$, use the principle of stationery potential energy to find the maximum deflection of the beam.



Solution:

The potential energy of the system is given by

$\Pi = \mathrm{U} - \mathrm{W}$

The strain energy in the beam is

$$\mathrm{U} = \int_V \frac{1}{2} \sigma_x \in_x \mathrm{dV}$$

Since $\sigma_x = \frac{My}{I}$ and correspondingly $\in_x = \frac{My}{EI}$,

$$= \int_V \frac{1}{2} \frac{My}{I} \frac{My}{EI} dV$$

$$= \int_V \frac{1}{2} \frac{M^2 y^2}{EI^2} dV$$

Now given,

$$M = EI \frac{d^2 v}{dx^2}$$

$$U = \int_V \frac{1}{2} \frac{E^2 I^2 y^2}{EI^2} \left(\frac{d^2 v}{dx^2} \right)^2 dV$$

$$\int_V \frac{1}{2} E \left(\frac{d^2 v}{dx^2} \right)^2 y^2 dV$$

$$= \int_0^L \frac{1}{2} E \left(\frac{d^2 v}{dx^2} \right)^2 \int y^2 dA\, dx$$

But $I = \int y^2 dA$, hence

$$U = \int_0^L \frac{1}{2} E\, I \left(\frac{d^2 v}{dx^2} \right)^2 dx$$

$$W = -\int_0^L pv\, dx$$

$$\Pi = U - W$$

$$= \int_0^L \left[\frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 + pv \right] dx$$

$$v = a_1 x (L - x)$$

$$\frac{dv}{dx} = a_1 L - 2a_1 x$$

$$\frac{d^2 v}{dx^2} = -2a_1$$

Substituting expressions for 'v' and $\frac{'d^2 v'}{dx^2}$

$$\Pi = \int_0^L \left[\frac{EI}{2} (-2a_1)^2 + pa_1 x (L - x) \right] dx$$

$$= 2\, E I a_1^2 L + pa_1 \frac{L^3}{6}$$

$$\frac{d\Pi}{da_1} = 0$$

$$4 E I a_1 L + p \frac{L^3}{6} = 0$$

$$a_1 = -\frac{pL^2}{24EI}$$

$$v = -\frac{pL^2}{24EI} x (L - x)$$

Maximum displacement would be at $x = \frac{L}{2}$

$$v_{\max} = -\frac{pL^4}{96EI}$$

$$\left(v_{\max}\right)_{exact} = -\frac{pL^4}{76.8EI}$$

From mechanics of materials solution.

The absolute relative true error is

$$\left|\in_t\right| = \left|\frac{-\frac{pL^4}{76.8EI} + \frac{pL^4}{96EI}}{-\frac{pL^4}{76.8EI}}\right| \times 100$$

$$= 17\%$$

Additional Exercise:

Rework the problem by claming the admissible displacement to be

$$v = a_1 x(L-x) + a_2 x^2 (L-x)^2$$

Can you find the <u>expression</u> for displacement, bending moment and bending stresses? Compare it to the exact value from the mechanics of materials solution.

Answer: You will find $a_1 = -\frac{pL^2}{24EI}$, $a_2 = -\frac{p}{24EI}$,

$$v = -\frac{pL^2}{24EI}x(L-x) - \frac{p}{24EI}x^2(L-x)^2$$

$$v_{\max} = -\frac{pL^4}{76.8EI}$$