

# Chapter

## A Primer on Matrix Algebra

### What is a matrix?

Matrices are everywhere. If you have used a spreadsheet such as Excel or Lotus or written a table, you have used a matrix. Matrices make presentation of numbers clearer and make calculations easier to program. Look at the matrix below about the sale of tires in a Blowout's store – given by quarter and make of tires.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Tirestone	25	20	3	2
Michigan	5	10	15	25
Copper	6	16	7	27

If one wants to know how many Copper tires were sold in Quarter 4, we go along the row 'Copper' and column 'Quarter 4' and find that it is 27.

### So what is a matrix?

A matrix is a rectangular array of elements. The elements can be symbolic expressions or numbers. Matrix [A] is denoted by

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Row i of [A] has n elements and is  $[a_{i1} \ a_{i2} \dots a_{in}]$  and

Column j of [A] has m elements and is  $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

Each matrix has rows and columns and this defines the size of the matrix. If a matrix [A] has m rows and n columns, the **size of the matrix** is denoted by m x n. The matrix [A] may also be denoted by  $[A]_{m \times n}$  to show that [A] is a matrix with m rows and n columns.

Each entry in the matrix is called the **entry** or **element** of the matrix and is denoted by  $a_{ij}$  where i is the row number and j is the column number of the element.

The matrix for the tire sales example could be denoted by the matrix [A] as

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

There are 3 rows and 4 columns, so the size of the matrix is 3 x 4. In the above [A] matrix,  $a_{34} = 27$ .

### What are the special types of matrices?

**Vector:** A vector is a matrix that has only one row or one column. There are two types of vectors – row vectors and column vectors.

**Row vector:** If a matrix has one row, it is called a row vector

$$[B] = [b_1 \ b_2 \ \dots \ b_m]$$

and 'm' is the dimension of the row vector.

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### Example

Give an example of a row vector.

### Solution

$[B] = [25 \ 20 \ 3 \ 2 \ 0]$  is an example of a row vector of dimension 5.

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**Column vector:** If a matrix has one column, it is called a column vector

$$[C] = \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_n \end{bmatrix}$$

and n is the dimension of the vector.

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### Example

Give an example of a column vector.

**Solution**

$$[C] = \begin{bmatrix} 25 \\ 5 \\ 6 \end{bmatrix} \text{ is an example of a column vector}$$

of dimension 3.

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**Square matrix:** If the number of rows ( $m$ ) of a matrix is equal to the number of columns ( $n$ ) of the matrix, ( $m = n$ ), it is called a square matrix. The entries  $a_{11}, a_{22}, \dots, a_{nn}$  are called the **diagonal elements** of a square matrix. Sometimes the diagonal of the matrix is also called the **principal** or **main** of the matrix.

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**Example**

Give an example of a square matrix.

**Solution**

$$[A] = \begin{bmatrix} 25 & 20 & 3 \\ 5 & 10 & 15 \\ 6 & 15 & 7 \end{bmatrix}$$

is a square matrix as it has same number of rows and columns, that is, three.

The diagonal elements of  $[A]$  are  $a_{11} = 25, a_{22} = 10, a_{33} = 7$ .

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**Upper triangular matrix:** A  $m \times n$  matrix for which  $a_{ij} = 0, i > j$  is called an upper triangular matrix. That is, all the elements below the diagonal entries are zero.

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**Example**

Give an example of an upper triangular matrix.

**Solution**

$$[A] = \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix}$$

is an upper triangular matrix.

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**Lower triangular matrix:** A  $m \times n$  matrix for which  $a_{ij} = 0, j > i$  is called a lower triangular matrix. That is, all the elements above the diagonal entries are zero.

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**Example**

Give an example of a lower triangular matrix.

**Solution**

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ 0.6 & 2.5 & 1 \end{bmatrix}$$

is a lower triangular matrix.

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**Diagonal matrix:** A square matrix with all non-diagonal elements equal to zero is called a diagonal matrix, that is, only the diagonal entries of the square matrix can be non-zero, ( $a_{ij} = 0, i \neq j$ ).

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**Example**

Give examples of a diagonal matrix.

**Solution**

$$[A] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

is a diagonal matrix.

Any or all the diagonal entries of a diagonal matrix can be zero.

For example

$$[A] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is also a diagonal matrix.

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**Identity matrix:** A diagonal matrix with all diagonal elements equal to one is called an identity matrix, ( $a_{ij} = 0$ ,  $i \neq j$ ; and  $a_{ii} = 1$  for all  $i$ ).

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### Example

Give an example of an identity matrix.

### Solution

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an identity matrix.

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**Zero matrix:** A matrix whose all entries are zero is called a zero matrix, ( $a_{ij} = 0$  for all  $i$  and  $j$ ).

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### Example

Give examples of a zero matrix.

### Solution

$$[A] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[D] = [0 \ 0 \ 0]$$

are all examples of a zero matrix.

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**Tridiagonal matrices:** A tridiagonal matrix is a square matrix in which all elements not on the following are zero: major diagonal, the diagonal above the major diagonal and the diagonal below the major diagonal.

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### Example

Give an example of a tridiagonal matrix.

### Solution

$$[A] = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 2 & 3 & 9 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

is a tridiagonal matrix.

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### Do non-square matrices have diagonal entries?

Yes, for a  $m \times n$  matrix  $[A]$ , the diagonal entries are  $a_{11}, a_{22}, \dots, a_{k-1,k-1}, a_{kk}$  where  $k = \min\{m, n\}$ .

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### Example

What are the diagonal entries of

$$[A] = \begin{bmatrix} 3.2 & 5 \\ 6 & 7 \\ 2.9 & 3.2 \\ 5.6 & 7.8 \end{bmatrix}$$

## Solution

The diagonal elements of [A] are  $a_{11} = 3.2$  and  $a_{22} = 7$ .

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**Diagonally Dominant Matrix:** A nxn square matrix [A] is a diagonally dominant matrix if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ i \neq j}}^n |a_{ij}| \text{ for all } i = 1, 2, \dots, n$$

$$\text{and } |a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^n |a_{ij}| \text{ for at least one } i,$$

that is, for each row, the absolute value of the diagonal element is greater than or equal to the sum of the absolute values of the rest of the elements of that row, and that the inequality is strictly greater than for at least one row. Diagonally dominant matrices are important in ensuring convergence in iterative schemes of solving simultaneous linear equations.

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## Example

Give examples of diagonally dominant matrices and not diagonally dominant matrices.

## Solution

$$[A] = \begin{bmatrix} 15 & 6 & 7 \\ 2 & -4 & -2 \\ 3 & 2 & 6 \end{bmatrix}$$

is a diagonally dominant matrix

as

$$|a_{11}| = |15| = 15 \geq |a_{12}| + |a_{13}| = |6| + |7| = 13$$

$$|a_{22}| = |-4| = 4 \geq |a_{21}| + |a_{23}| = |2| + |2| = 4$$

$$|a_{33}| = |6| = 6 \geq |a_{31}| + |a_{32}| = |3| + |2| = 5$$

and for at least one row, that is Rows 1 and 3 in this case, the inequality is a strictly greater than inequality.

$$[A] = \begin{bmatrix} -15 & 6 & 9 \\ 2 & -4 & 2 \\ 3 & -2 & 5.001 \end{bmatrix}$$

is a diagonally dominant matrix

as

$$|a_{11}| = |-15| = 15 \geq |a_{12}| + |a_{13}| = |6| + |9| = 15$$

$$|a_{22}| = |-4| = 4 \geq |a_{21}| + |a_{23}| = |2| + |2| = 4$$

$$|a_{33}| = |5.001| = 5.001 \geq |a_{31}| + |a_{32}| = |3| + |-2| = 5$$

the inequalities are satisfied for all rows and it is satisfied strictly greater than for at least one row (in this case it is Row 3)

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

is not diagonally dominant as

$$|a_{22}| = |8| = 8 \leq |a_{21}| + |a_{23}| = |64| + |1| = 65$$

### When are two matrices considered to be equal?

Two matrices  $[A]$  and  $[B]$  are equal if the size of  $[A]$  and  $[B]$  is the same (number of rows and columns are same for  $[A]$  and  $[B]$ ) and  $a_{ij} = b_{ij}$  for all  $i$  and  $j$ .

### Example

What would make

$$[A] = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \text{ to be equal to}$$

$$[B] = \begin{bmatrix} b_{11} & 3 \\ 6 & b_{22} \end{bmatrix},$$

### Solution

The two matrices  $[A]$  and  $[B]$  would be equal if

$$b_{11} = 2, b_{22} = 7.$$

**Symmetric matrix:** A square matrix  $[A]$  with real elements where  $a_{ij} = a_{ji}$  for  $i=1, \dots, n$  and  $j=1, \dots, n$  is called a symmetric matrix. This is same as, if  $[A] = [A]^T$ , then  $[A]$  is a symmetric matrix.

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### Example

Give an example of a symmetric matrix.

### Solution

$$[A] = \begin{bmatrix} 21.2 & 3.2 & 6 \\ 3.2 & 21.5 & 8 \\ 6 & 8 & 9.3 \end{bmatrix}$$

is a symmetric matrix as  $a_{12} = a_{21} = 3.2$ ;  $a_{13} = a_{31} = 6$  and  $a_{23} = a_{32} = 8$ .

### How do you add two matrices?

Two matrices  $[A]$  and  $[B]$  can be added only if they are the same size, then the addition is shown as

$$[C] = [A] + [B]$$

where

$$c_{ij} = a_{ij} + b_{ij}$$

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### Example

Add two matrices

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$
$$[B] = \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

### Solution

$$[C] = [A] + [B]$$

$$\begin{aligned}
&= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix} \\
&= \begin{bmatrix} 5+6 & 2+7 & 3-2 \\ 1+3 & 2+5 & 7+19 \end{bmatrix} \\
&= \begin{bmatrix} 11 & 9 & 1 \\ 4 & 7 & 26 \end{bmatrix}
\end{aligned}$$

### Example

Blowout r'us store has two locations 'A' and 'B', and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

where the rows represent sale of Tirestone, Michigan and Copper tires and the columns represent the quarter number - 1, 2, 3, 4. What are the total sales of the two locations by make and quarter?

### Solution

$$[C] = [A] + [B]$$

$$\begin{aligned}
&= \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} + \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix} \\
&= \begin{bmatrix} (25+20) & (20+5) & (3+4) & (2+0) \\ (5+3) & (10+6) & (15+15) & (25+21) \\ (6+4) & (16+1) & (7+7) & (27+20) \end{bmatrix} \\
&= \begin{bmatrix} 45 & 25 & 7 & 2 \\ 8 & 16 & 30 & 46 \\ 10 & 17 & 14 & 47 \end{bmatrix}
\end{aligned}$$

So if one wants to know the total number of Copper tires sold in quarter 4 in the two locations, we would look at Row 3 – Column 4 to give

$$c_{34} = 47.$$

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### How do you subtract two matrices?

Two matrices [A] and [B] can be subtracted only if they are the same size and the subtraction is given by

$$[D] = [A] - [B]$$

where

$$d_{ij} = a_{ij} - b_{ij}$$

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### Example

Subtract matrix [B] from matrix [A].

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

### Solution

$$\begin{aligned} [C] &= [A] - [B] \\ &= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix} \\ &= \begin{bmatrix} 5-6 & 2-7 & 3-(-2) \\ 1-3 & 2-5 & 7-19 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -5 & 5 \\ -2 & -3 & -12 \end{bmatrix} \end{aligned}$$

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### Example

Blowout r'us store has two locations A and B and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix}$$

where the rows represent sale of Tirestone, Michigan and Copper tires and the columns represent the quarter number- 1, 2, 3, 4. How many more tires did store A sell than store B of each brand in each quarter?

### Solution

$$\begin{aligned} & [D] \\ & = [A] - [B] \\ & = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} - \begin{bmatrix} 20 & 5 & 4 & 0 \\ 3 & 6 & 15 & 21 \\ 4 & 1 & 7 & 20 \end{bmatrix} \\ & = \begin{bmatrix} 25-20 & 20-5 & 3-4 & 2-0 \\ 5-3 & 10-6 & 15-15 & 25-21 \\ 6-4 & 16-1 & 7-7 & 27-20 \end{bmatrix} \\ & = \begin{bmatrix} 5 & 15 & -1 & 2 \\ 2 & 4 & 0 & 4 \\ 2 & 15 & 0 & 7 \end{bmatrix} \end{aligned}$$

So if you want to know how many more Copper Tires were sold in quarter 4 in Store A than Store B,  $d_{34} = 7$ . Note that  $d_{13} = -1$  implies that store A sold 1 less Michigan tire than Store B in quarter 3.

### How do I multiply two matrices?

Two matrices [A] and [B] can be multiplied only if the number of columns of [A] is equal to the number of rows of [B] to give

$$[C]_{m \times n} = [A]_{m \times p} [B]_{p \times n}$$

If [A] is a  $m \times p$  matrix and [B] is a  $p \times n$  matrix, the resulting matrix [C] is a  $m \times n$  matrix.

So how does one calculate the elements of [C] matrix?

$$c_{ij} = \sum_{k=1}^p a_{ik} b_{kj}$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ip}b_{pj}$$

for each  $i = 1, 2, \dots, m$ , and  $j = 1, 2, \dots, n$ .

To put it in simpler terms, the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of the [C] matrix in  $[C] = [A][B]$  is calculated by multiplying the  $i^{\text{th}}$  row of [A] by the  $j^{\text{th}}$  column of [B], that is,

$$c_{ij} = [a_{i1} \ a_{i2} \ \dots \ a_{ip}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix}$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{ip} b_{pj}$$

$$= \sum_{k=1}^p a_{ik} b_{kj}$$

### Example

Given

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 3 & -2 \\ 5 & -8 \\ 9 & -10 \end{bmatrix}$$

find

$$[C] = [A][B]$$

### Solution

$c_{12}$  can be found by multiplying the first row of [A] by the second column of [B],

$$c_{12} = [5 \ 2 \ 3] \begin{bmatrix} -2 \\ -8 \\ -10 \end{bmatrix}$$

$$= (5)(-2) + (2)(-8) + (3)(-10)$$

$$= -56$$

Similarly, one can find the other elements of [C] to give

$$[C] = \begin{bmatrix} 52 & -56 \\ 76 & -88 \end{bmatrix}$$

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### Example

Blowout r'us store location A and the sales of tires are given by make (in rows) and quarters (in columns) as shown below

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

where the rows represent sale of Tirestone, Michigan and Copper tires and the columns represent the quarter number - 1, 2, 3, 4. Find the per quarter sales of store A if following are the prices of each tire.

Tirestone = \$33.25

Michigan = \$40.19

Copper = \$25.03

### Solution

The answer is given by multiplying the price matrix by the quantity sales of store A. The price matrix is  $[33.25 \ 40.19 \ 25.03]$ , then the per quarter sales of store A would be given by

$$[C] = [33.25 \ 40.19 \ 25.03] \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^3 a_{ik} b_{kj}$$

$$c_{11} = \sum_{k=1}^3 a_{1k} b_{k1}$$

$$= a_{11} b_{11} + a_{12} b_{21} + a_{13} b_{31}$$

$$= (33.25)(25) + (40.19)(5) + (25.03)(6)$$

$$= \$1182.38$$

Similarly

$$c_{12} = \$1467.38,$$

$$c_{13} = \$877.81,$$

$$c_{14} = \$1747.06.$$

So each quarter sales of store A in dollars are given by the four columns of the row vector

$$[C] = [1182.38 \quad 1467.38 \quad 877.81 \quad 1747.06]$$

Remember since we are multiplying a 1 x 3 matrix by a 3 x 4 matrix, the resulting matrix is a 1 x 4 matrix.

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### What is a scalar product of a constant and a matrix?

If  $[A]$  is a  $n \times n$  matrix and  $k$  is a real number, then the scalar product of  $k$  and  $[A]$  is another matrix  $[B]$ , where  $b_{ij} = k a_{ij}$ .

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### Example

Let  $[A] = \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}$ . Find  $2[A]$

### Solution

$$[A] = \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}$$

Then

$$\begin{aligned} 2[A] &= 2 \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix} \\ &= \begin{bmatrix} (2)(2.1) & (2)(3) & (2)(2) \\ (2)(5) & (2)(1) & (2)(6) \end{bmatrix} \\ &= \begin{bmatrix} 4.2 & 6 & 4 \\ 10 & 2 & 12 \end{bmatrix} \end{aligned}$$

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