EML 4230 Introduction to Composite Materials

Chapter 2 Macromechanical Analysis of a Lamina
Review of Definitions

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Courtesy of the Textbook
Mechanics of Composite Materials by Kaw

UNIVERSITY OF SOUTH FLORIDA

Stress

$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta P_n}{\Delta A},$$

$$\tau_s = \lim_{\Delta A \to 0} \frac{\Delta P_s}{\Delta A}$$

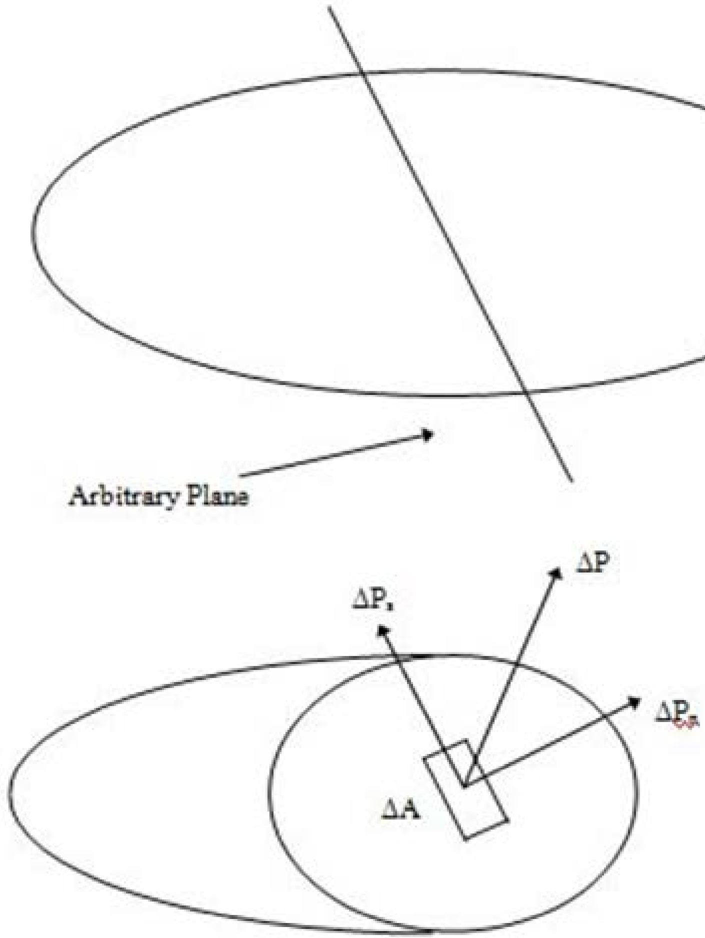

FIGURE 2.5
Stresses on infinitesimal area on an arbitrary plane

Stress

$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta P_x}{\Delta A},$$

$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta P_y}{\Delta A},$$

$$\tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta P_z}{\Delta A}$$

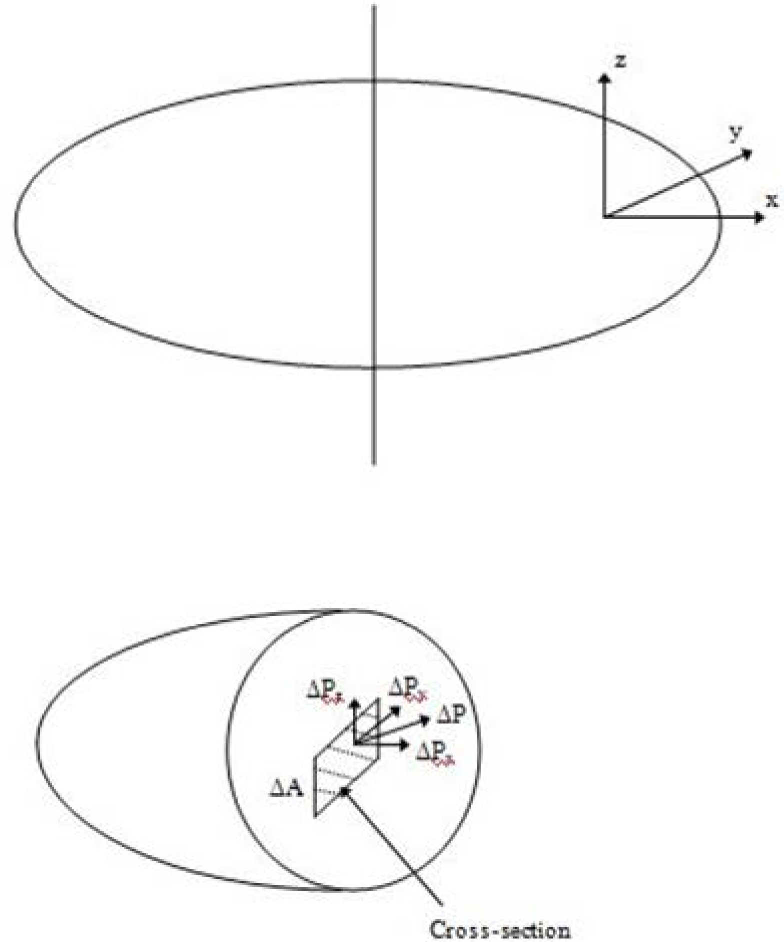

FIGURE 2.6
Forces on an infinitesimal area on the y-z plane

Stress

$$\tau_{xy} = \tau_{yx},$$

$$\tau_{yz} = \tau_{zy},$$

$$\tau_{zx} = \tau_{xz}$$

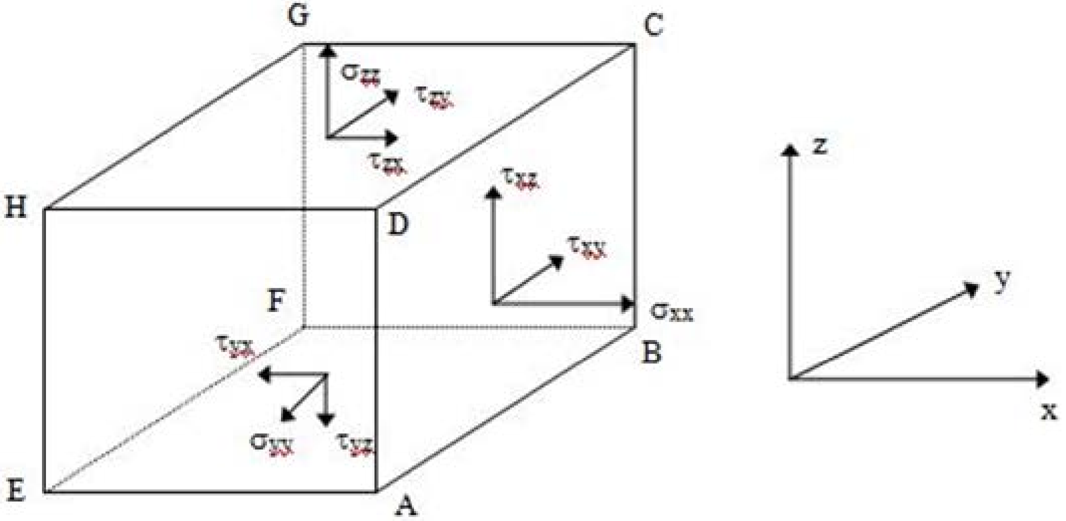

FIGURE 2.7
Stresses on an infinitesimal cuboid

Strain

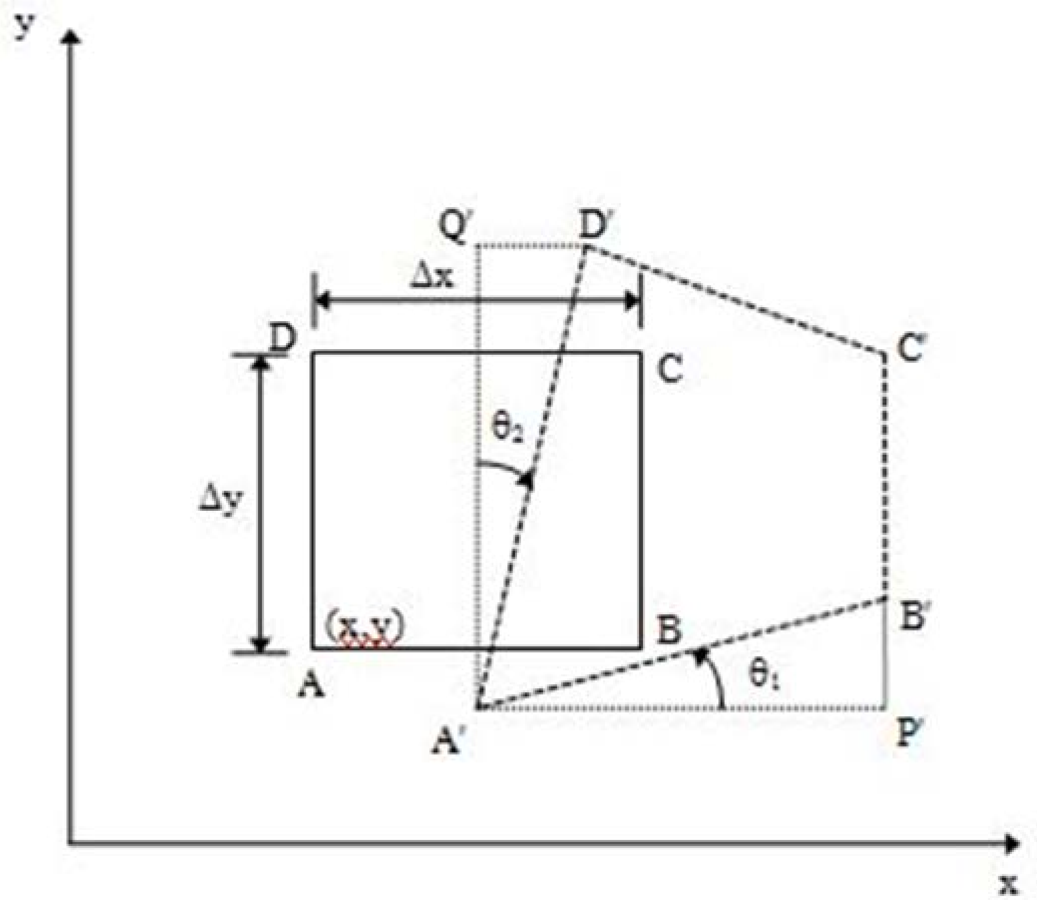

FIGURE 2.8
Normal and shearing strains on an infinitesimal area in the x-y plane

Strain

$u = u(x,y,z)$ = displacement in x-direction at point (x,y,z),

$v = v(x,y,z)$ = displacement in y-direction at point (x,y,z),

$w = w(x,y,z)$ = displacement in z-direction at point (x,y,z)

$$\varepsilon_x = \lim_{AB \to 0} \frac{A'B' - AB}{AB}$$

Where:

$$A'B' = \sqrt{(A'P')^2 + (B'P')^2}$$

$$= \sqrt{[\Delta x + u(x+\Delta x, y) - u(x,y)]^2 + [v(x+\Delta x, y) - v(x,y)]^2},$$

$$AB = \Delta x$$

Strain

Substituting

$$\varepsilon_x = \lim_{\Delta x \to 0} \left\{ \left[1 + \frac{u(x + \Delta +) - u(x,y)}{\Delta x} \right]^2 + \left[\frac{v(x + \Delta +) - v(x,y)}{\Delta x} \right]^2 \right\}^{1/2} - 1$$

$$\downarrow$$

$$\varepsilon_x = \left[\left(1 + \frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right]^{1/2} - 1$$

$$\searrow$$

$$\frac{\partial u}{\partial x} << 1 \longrightarrow$$

$$\frac{\partial v}{\partial x} << 1 \longrightarrow$$

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

Strain

$u = u(x,y,z)$ = displacement in x-direction at point (x,y,z),

$v = v(x,y,z)$ = displacement in y-direction at point (x,y,z),

$w = w(x,y,z)$ = displacement in z-direction at point (x,y,z)

$$\varepsilon_y = \lim_{AD \to 0} \frac{A'D' - AD}{AD}$$

Where:

$$A'D' = \sqrt{(A'Q')^2 + (Q'D')^2}$$

$$= \sqrt{[\Delta y + v(x,y+\Delta,) - v(x,y)]^2 + [u(x,y+\Delta,) - u(x,y)]^2},$$

$$AD = \Delta y$$

Strain

Substituting

$$\varepsilon_y = \lim_{\Delta y \to 0} \left\{ \left[1 + \frac{v(x,y+\Delta y) - v(x,y)\}}{\Delta y} \right]^2 + \left[\frac{u(x,y+\Delta y) - u(x,y)}{\Delta y} \right]^2 \right\}^{1/2} - 1$$

↓

$$\varepsilon_y = \left[\left(1 + \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right]^{1/2} -1$$

$$\frac{\partial u}{\partial y} << 1 \longrightarrow$$

$$\frac{\partial v}{\partial y} << 1 \longrightarrow$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

Strain

$$\gamma_{xy} = \theta_1 + \theta_2$$

Where:

$$\theta_1 = \lim_{AB \to 0} \frac{P'B'}{A'P'},$$

$$P'B' = v(x + \Delta x, y) - v(x, y),$$

$$A'P' = u(x + \Delta x, y) + \Delta x - u(x, y)$$

Strain

$$\gamma_{xy} = \theta_1 + \theta_2$$

Where:

$$\theta_2 = \lim_{AD \to 0} \frac{Q'D'}{A'Q'},$$

$$Q'D' = u(x, y + \Delta y) - u(x, y),$$

$$A'Q' = v(x, y + \Delta y) + \Delta y - v(x, y)$$

Strain

Substituting

$$\gamma_{xy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\dfrac{v(x+\Delta x, y) - v(x,y)}{\Delta x}}{\dfrac{u(x+\Delta x, y) + \Delta x - u(x,y)}{\Delta x}} + \frac{\dfrac{u(x, y+\Delta y) - u(x,y)}{\Delta y}}{\dfrac{v(x, y+\Delta y) + \Delta y - v(x,y)}{\Delta y}}$$

↓

$$\gamma_{xy} = \frac{\dfrac{\partial v}{\partial x}}{1 + \dfrac{\partial u}{\partial x}} + \frac{\dfrac{\partial u}{\partial y}}{1 + \dfrac{\partial u}{\partial y}}$$

$$\frac{\partial u}{\partial x} << 1 \longrightarrow$$

$$\frac{\partial v}{\partial y} << 1 \longrightarrow$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Strain

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z},$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$

Elastic Moduli

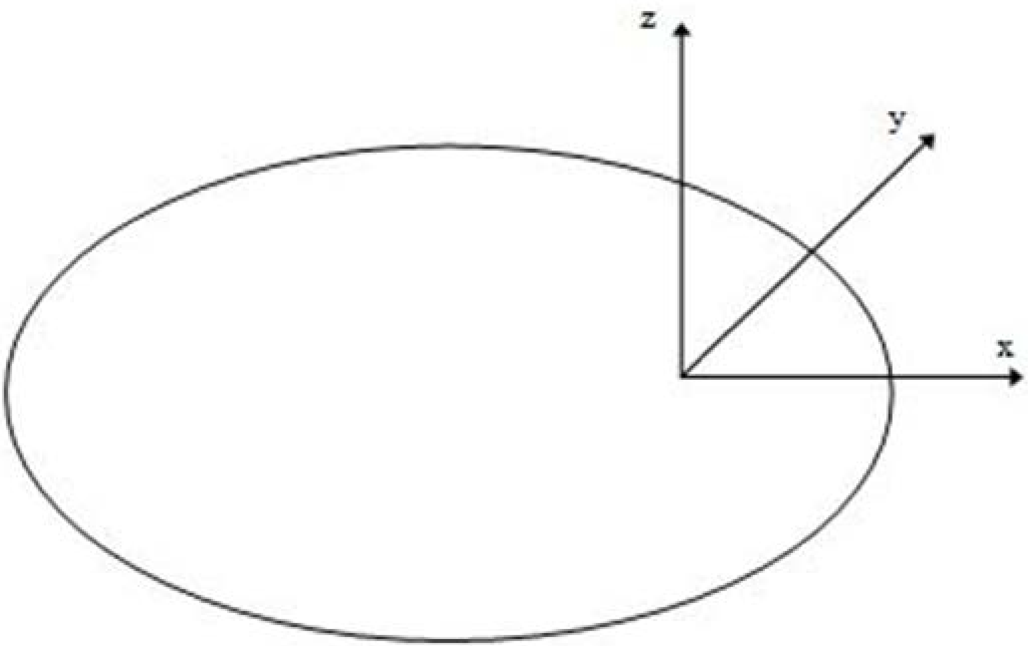

FIGURE 2.9
Cartesian coordinates in 3-D

Elastic Moduli

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{v}{E} & -\frac{v}{E} & 0 & 0 & 0 \\ -\frac{v}{E} & \frac{1}{E} & -\frac{v}{E} & 0 & 0 & 0 \\ -\frac{v}{E} & -\frac{v}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}$$

Elastic Moduli

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} =
\begin{bmatrix}
\frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
\frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
\frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
0 & 0 & 0 & G & 0 & 0 \\
0 & 0 & 0 & 0 & G & 0 \\
0 & 0 & 0 & 0 & 0 & G
\end{bmatrix}
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix}
$$

Where:

$$G = \frac{E}{2(1+\nu)}$$

Strain Energy

$$W = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx})$$

END