EML 4230 Introduction to Composite Materials

Chapter 3 Micromechanical Analysis of a Lamina
Elastic Moduli of Unidirectional Lamina
Longitudinal Young's Modulus

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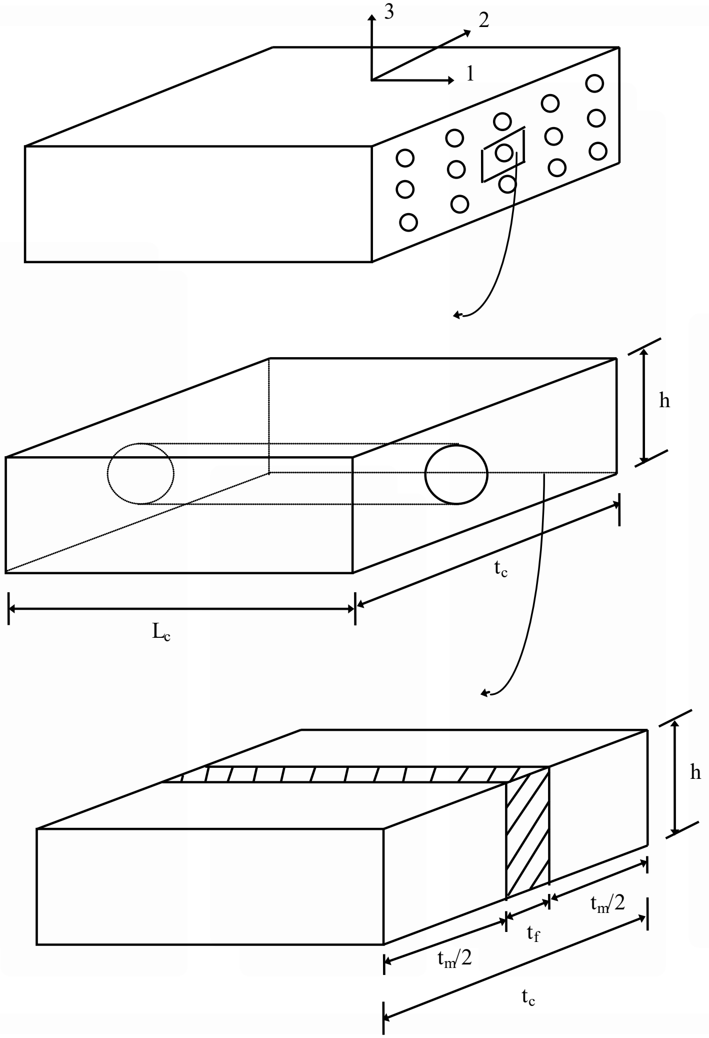
Courtesy of the Textbook
Mechanics of Composite Materials by Kaw

UNIVERSITY OF SOUTH FLORIDA

Elastic Moduli of Unidirectional Lamina

- Longitudinal elastic modulus, E_1
- Transverse elastic modulus, E_2
- Major Poisson's ratio, ν_{12}
- In-plane shear modulus, G_{12}

Strength of Materials Approach

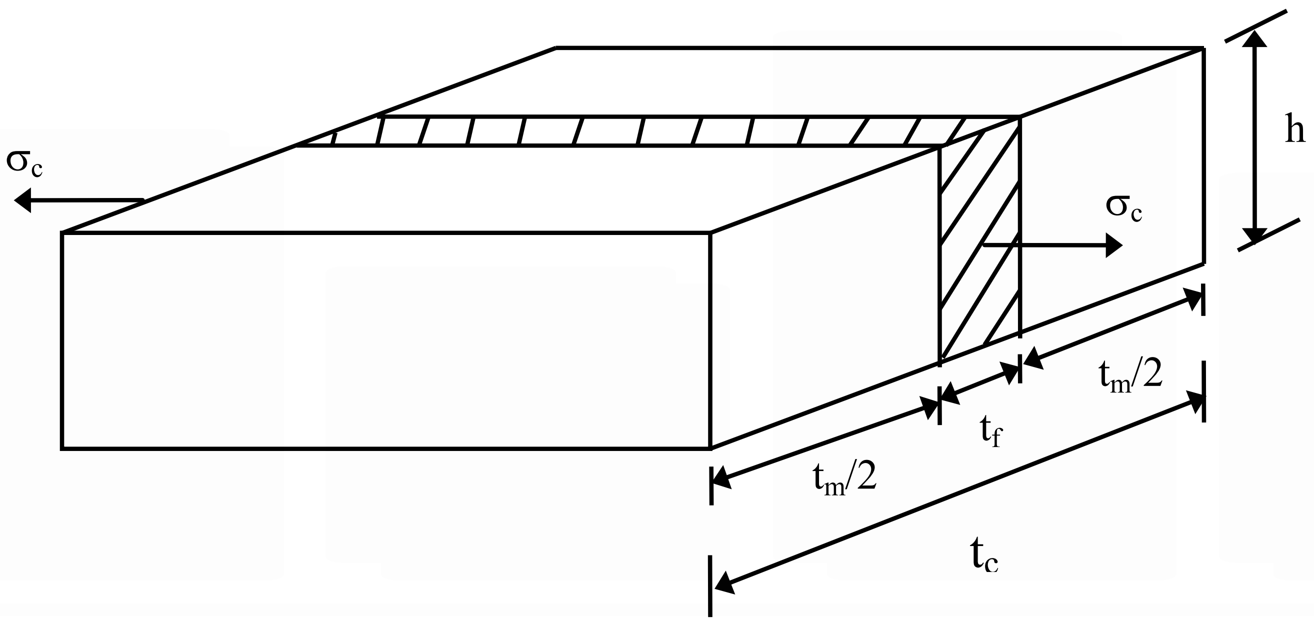


$$V_f = \frac{h t_f L_c}{h\, t_c L_c} = \frac{t_f}{t_c}$$

$$V_m = \frac{h t_m L_c}{h\, t_c L_c} = \frac{t_m}{t_c}$$

FIGURE 3.3
Representative volume element of a unidirectional lamina.

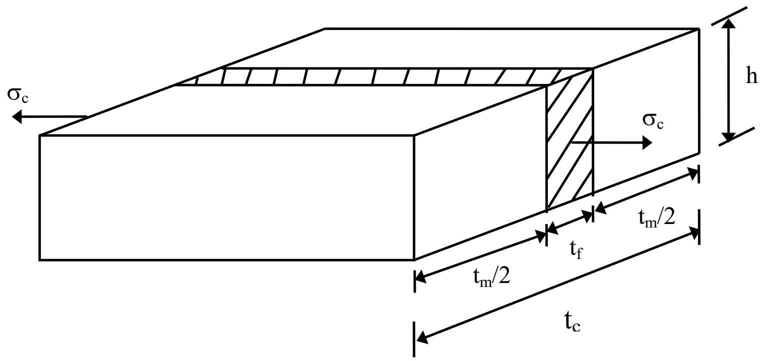
Longitudinal Young's Modulus, E1



A uniform longitudinal strain applied to the representative volume element to calculate the longitudinal Young's modulus for a unidirectional lamina.

Longitudinal Young's Modulus

$$F_c = F_f + F_m$$



$$F_c = \sigma_c A_c,$$

$$F_f = \sigma_f A_f, \text{ and}$$

$$F_m = \sigma_m A_m$$

$$\sigma_c = E_1 \varepsilon_c,$$

$$\sigma_f = E_f \varepsilon_f, \text{ and}$$

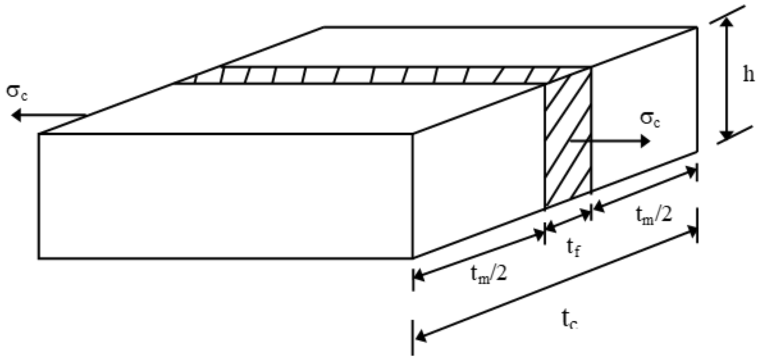
$$\sigma_m = E_m \varepsilon_m$$

Longitudinal Young's Modulus

$$F_c = F_f + F_m$$

$$E_1 \varepsilon_c A_c = E_f \varepsilon_f A_f + E_m \varepsilon_m A_m$$

$$\text{If } (\varepsilon_c = \varepsilon_f = \varepsilon_m), \text{ then :}$$



$$E_1 = E_f \frac{A_f}{A_c} + E_m \frac{A_m}{A_c}$$

$$E_1 = E_f V_f + E_m V_m$$

Ratio of force taken by fiber to composite

$$F_c = \sigma_c A_c ,$$

$$F_f = \sigma_f A_f , \text{ and}$$

$$F_m = \sigma_m A_m$$

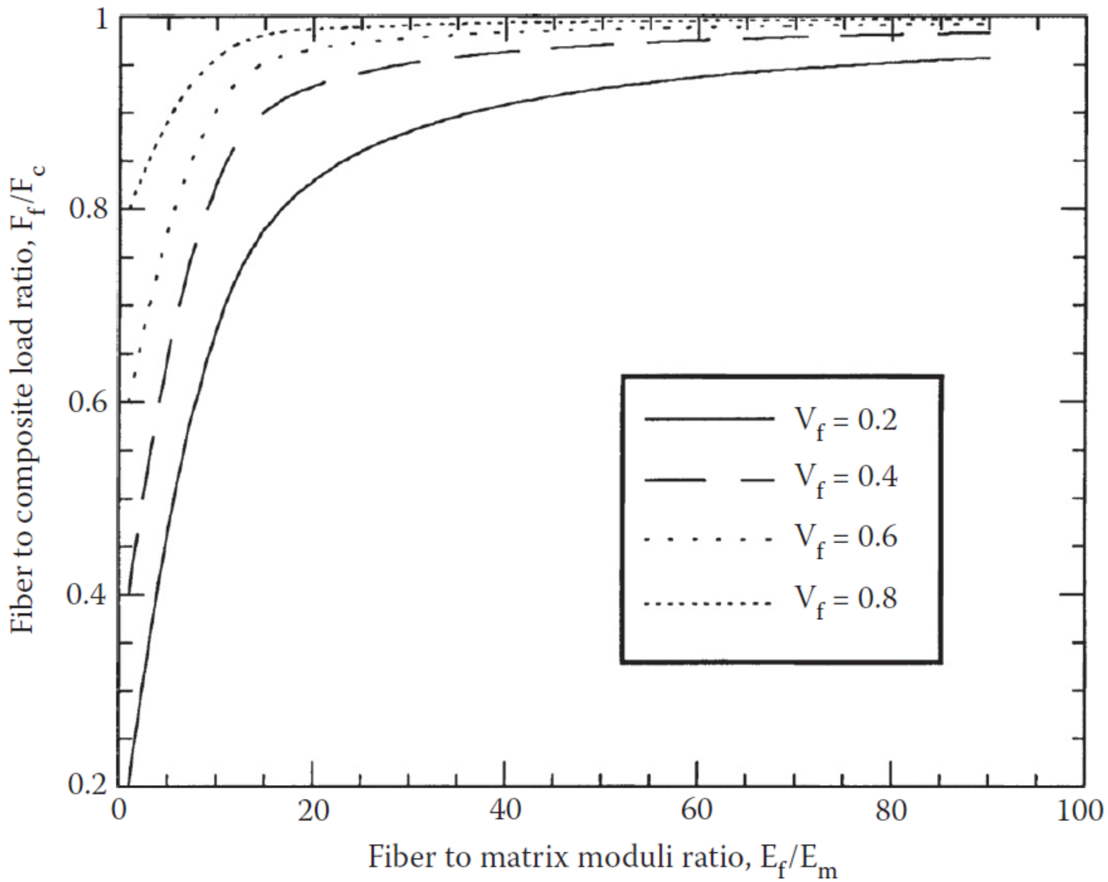
$$\sigma_c = E_1 \varepsilon_c ,$$

$$\sigma_f = E_f \varepsilon_f , \text{ and}$$

$$\sigma_m = E_m \varepsilon_m$$

$$\frac{F_f}{F_c} = \frac{E_f}{E_1} V_f$$

Ratio of force taken by fiber to composite



$$\frac{F_f}{F_c} = \frac{E_f}{E_1} V_f$$

FIGURE 3.5
Fraction of load of composite carried by fibers as a function of fiber volume fraction for constant fiber to matrix moduli ratio.

Example – Longitudinal Young's Modulus

Find the longitudinal elastic modulus of a unidirectional Glass/Epoxy lamina with a 70% fiber volume fraction. Use the properties of glass and epoxy from Tables 3.1 and 3.2, respectively. Also, find the ratio of the load the fibers take to that of the composite.

$$E_f = 85\ GPa$$

$$E_m = 3.4\ GPa$$

$$E_1 = E_f V_f + E_m V_m$$

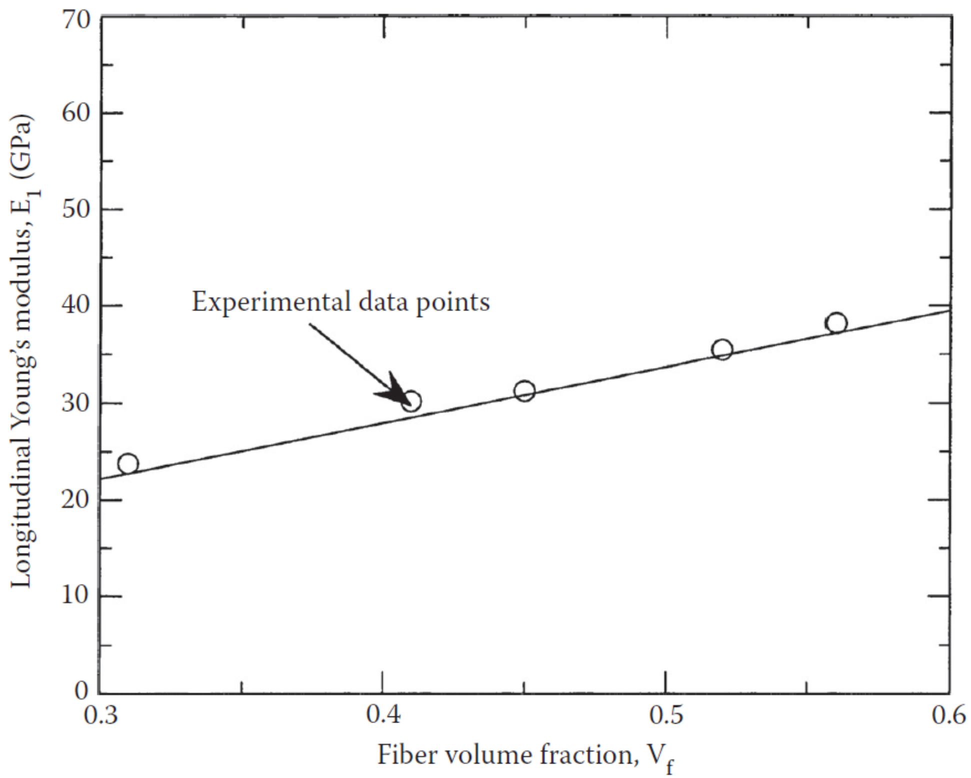
$$E_1 = (85)(0.7) + (3.4)(0.3)$$
$$= 60.52\ GPa$$

Force in fiber to composite ratio

$$\frac{F_f}{F_c} = \frac{E_f}{E_1} V_f$$

$$\frac{F_f}{F_c} = \frac{85}{60.52}(0.7) = 0.9831$$

Comparing with experimental results



Longitudinal Young's modulus as a function of fiber volume fraction and comparison with experimental data points for a typical glass/polyester lamina

END

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Chapter 3 Micromechanical Analysis of a Lamina

Elastic Moduli of Unidirectional Lamina

Major Poisson's ratio, ν_{12}

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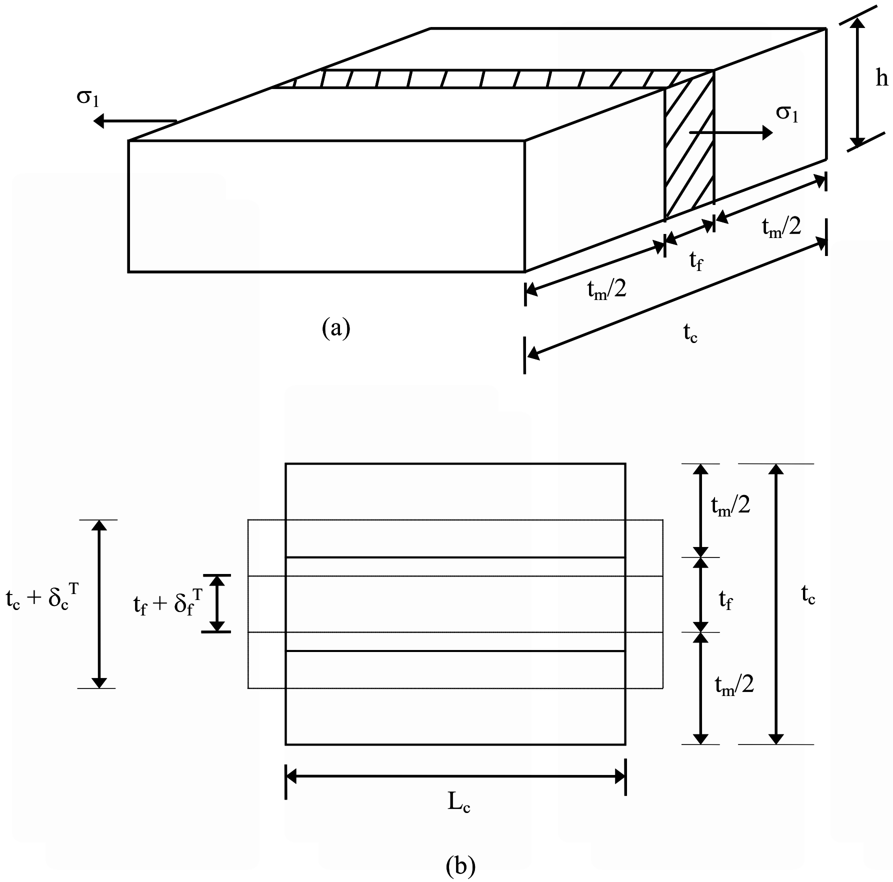
- Major Poisson's ratio, ν_{12}
- In-plane shear modulus, G_{12}

Major Poisson's ratio, ν_{12}



$$\nu_{12} = \nu_f V_f + \nu_m V_m$$

A longitudinal strain applied to a representative volume element to calculate Poisson's ratio of unidirectional lamina.

Example

Find the major and minor Poisson's ratio of a Glass/Epoxy lamina with a 70% fiber volume fraction. Use the properties of glass and epoxy from Tables 3.1 and 3.2, respectively.

$$\nu_f = 0.2$$

$$\nu_m = 0.3$$

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$

$$\nu_{12} = (0.2)(0.7) + (0.3)(0.3)$$
$$= 0.230$$

Example

$$E_1 = 60.52\ GPa$$

$$E_2 = 10.37\ GPa$$

$$\nu_{21} = \nu_{12} \frac{E_2}{E_1}$$

$$= 0.230 \frac{10.37}{60.52}$$

$$= 0.03941$$

END

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Chapter 3 Micromechanical Analysis of a Lamina
Elastic Moduli of Unidirectional Lamina
Transverse Young's Modulus, E2

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Elastic Moduli of Unidirectional Lamina

- Longitudinal elastic modulus, E_1
- Transverse elastic modulus, E_2
- Major Poisson's ratio, ν_{12}
- In-plane shear modulus, G_{12}

Transverse Young's Modulus, E_2

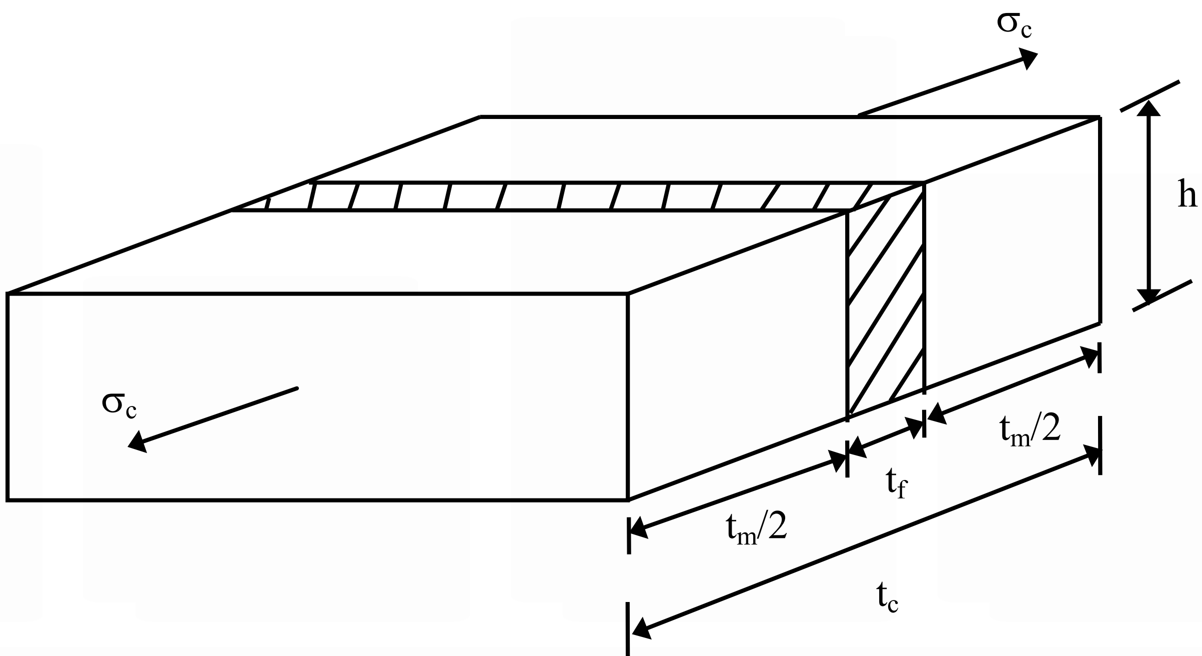


FIGURE 3.7
A transverse stress applied to a representative volume element used to calculate transverse Young's modulus of a unidirectional lamina.

Transverse Young's Modulus

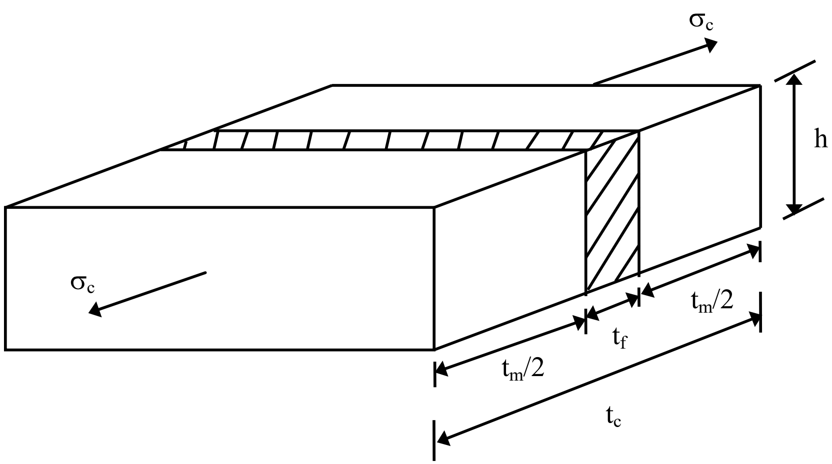
$$\sigma_c = \sigma_f = \sigma_m$$

$$\Delta_c = \Delta_f + \Delta_m$$

$$\Delta_c = t_c \varepsilon_c, \quad \varepsilon_c = \frac{\sigma_c}{E_2},$$

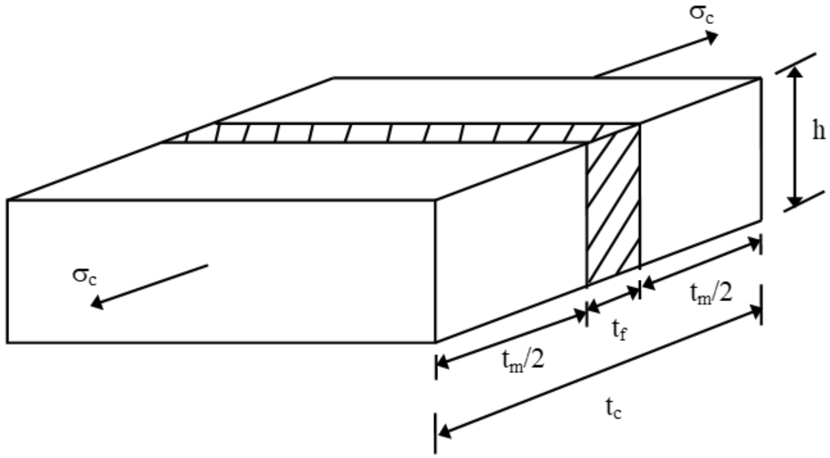
$$\Delta_f = t_f \varepsilon_f, \quad \varepsilon_f = \frac{\sigma_f}{E_f},$$

$$\Delta_m = t_m \varepsilon_m \quad \varepsilon_m = \frac{\sigma_m}{E_m}$$



$$\frac{t_c\sigma_c}{E_2} = \frac{t_f\sigma_f}{E_f} + \frac{t_m\sigma_m}{E_m}$$

Transverse Young's Modulus



$$\frac{t_c}{E_2} = \frac{t_f}{E_f} + \frac{t_m}{E_m}$$

$$\frac{1}{E_2} = \frac{1}{E_f}\frac{t_f}{t_c} + \frac{1}{E_m}\frac{t_m}{t_c}, and$$

$$\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$$

Example, Transverse Young's modulus

Find the transverse Young's modulus of a Glass/Epoxy lamina with a fiber volume fraction of 70%. Use the properties of glass and epoxy from Tables 3.1 and 3.2, respectively.

$$E_f = 85\ GPa$$

$$E_m = 3.4\ GPa$$

$$\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$$

$$\frac{1}{E_2} = \frac{0.7}{85} + \frac{0.3}{3.4}$$

$$E_2 = 10.37\ GPa$$

Transverse Young's Modulus vs Fiber Volume Fraction

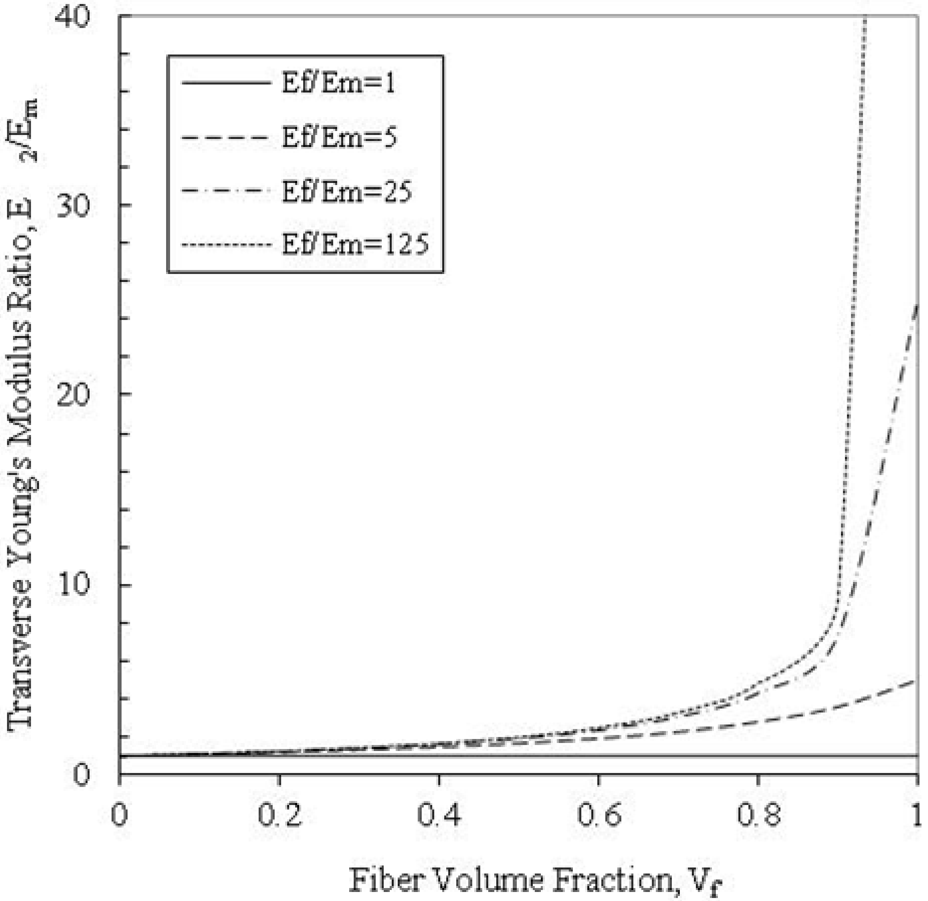
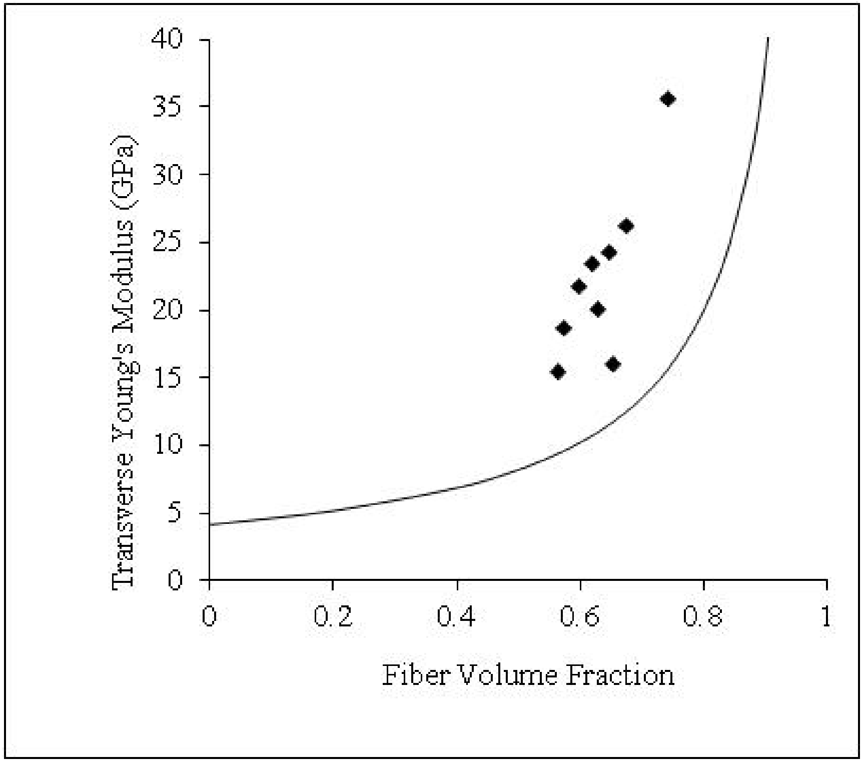


FIGURE 3.8
Transverse Young's modulus as a function of fiber volume fraction for constant fiber to matrix moduli ratio.

Transverse Young's modulus comparison with experimental data



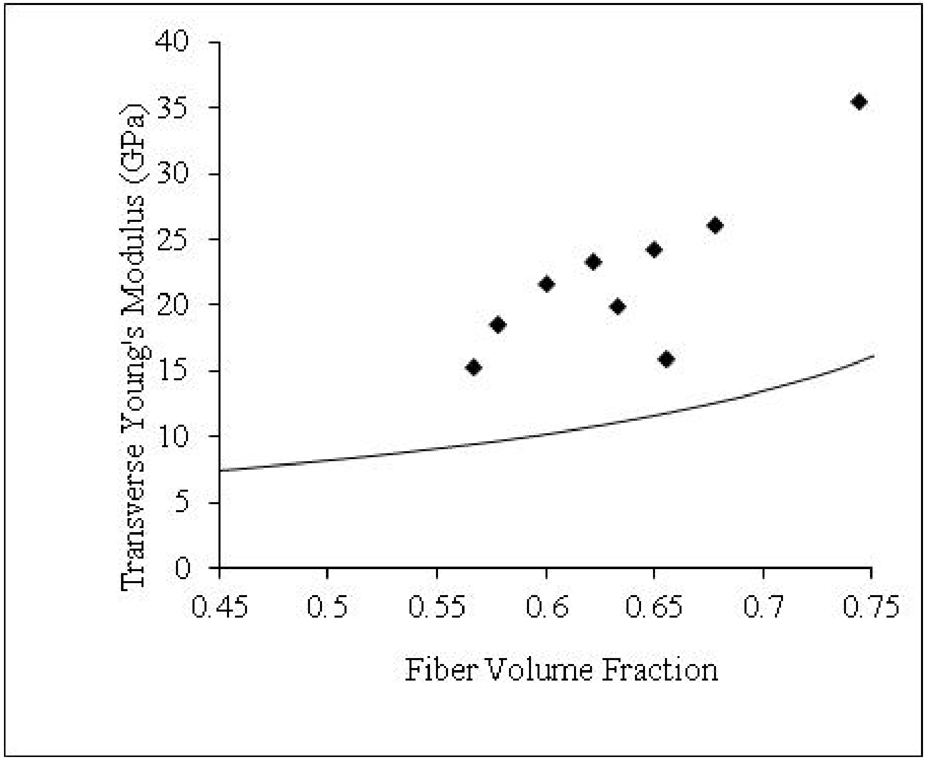


FIGURE 3.10 Theoretical values of transverse Young's modulus as a function of fiber volume fraction for a boron/epoxy unidirectional lamina (*Ef* = 414 GPa, *vf* = 0.2, *Em* = 4.14 GPa, *vm* = 0.35) and comparison with experimental values. Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract no. NAS1-8818, November 1970.)

END

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Elastic Moduli of Unidirectional Lamina
In-plane shear modulus, G_{12}

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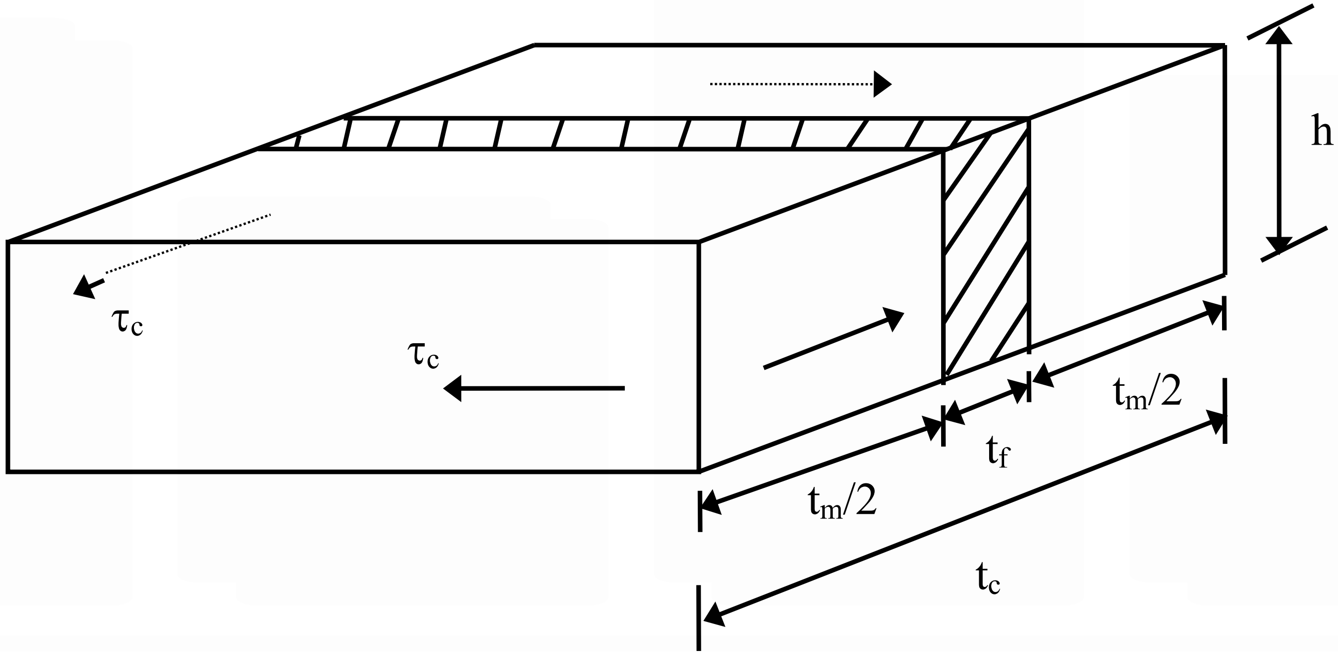
Elastic Moduli of Unidirectional Lamina

- Longitudinal elastic modulus, E_1
- Transverse elastic modulus, E_2
- Major Poisson's ratio, ν_{12}
- In-plane shear modulus, G_{12}

In-Plane Shear Modulus, G_{12}



An in-plane shear stress applied to a representative volume element for finding in-plane shear modulus of a unidirectional lamina.

$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}$$

Example, In-Plane Shear Modulus, G_{12}

Find the in-plane shear modulus of a Glass/Epoxy lamina with a 70% fiber volume fraction. Use properties of glass and epoxy from Tables 3.1 and 3.2, respectively.

$$E_f = 85\ GPa$$

$$\nu_f = 0.2$$

$$G_f = \frac{E_f}{2(1+\nu_f)}$$

$$= \frac{85}{2(1+0.2)}$$

$$= 35.42\ GPa$$

$$E_m = 3.4\ GPa$$

$$\nu_m = 0.3$$

$$G_m = \frac{E_m}{2(1+\nu_m)}$$

$$= \frac{3.40}{2(1+0.3)}$$

$$= 1.308\ GPa$$

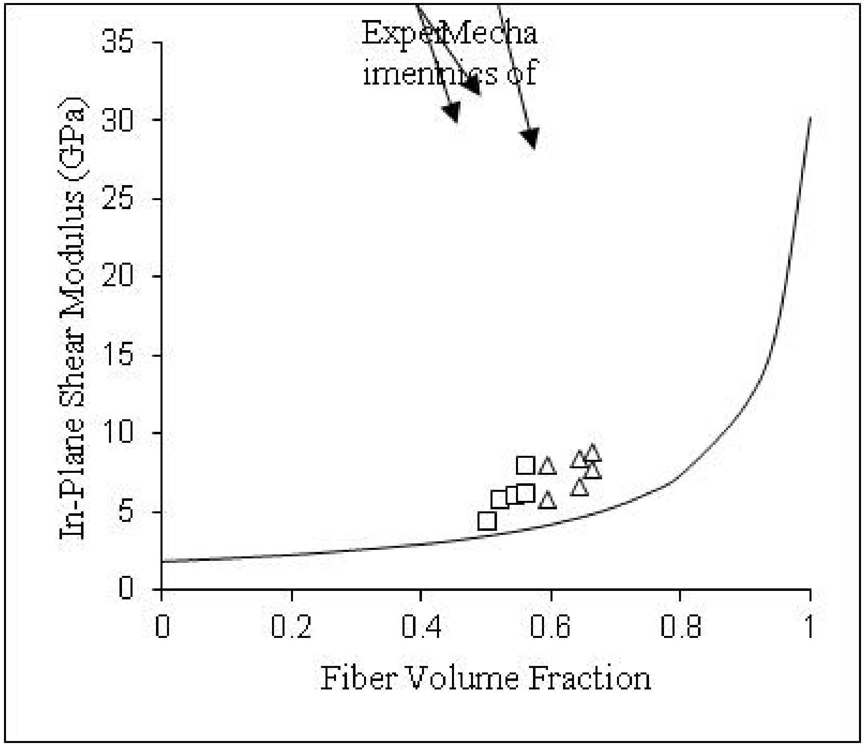
Example

$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}$$

$$\frac{1}{G_{12}} = \frac{0.70}{35.42} + \frac{0.30}{1.308}$$

$$G_{12} = 4.014\ GPa$$

In-Plane Shear Modulus comparison with experimental data



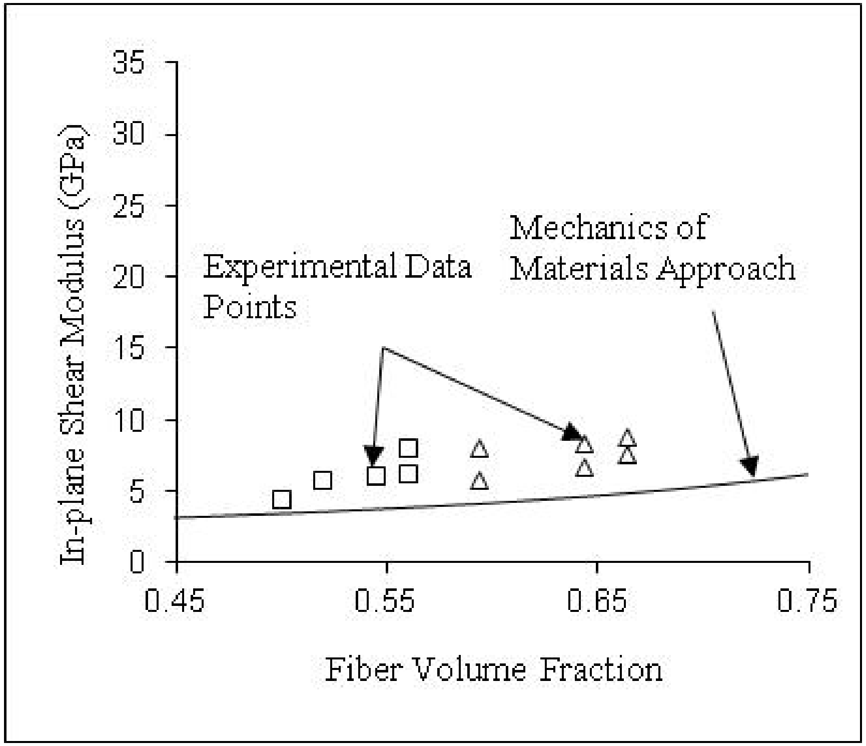


FIGURE 3.13 Theoretical values of in-plane shear modulus as a function of fiber volume fraction and comparison with experimental values for a unidirectional glass/epoxy lamina (*Gf* = 30.19 GPa, *Gm* = 1.83 GPa). Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract no. NAS1-8818, November 1970.)

Mechanics of Materials Equations for Elastic Moduli Summary

Longitudinal elastic modulus, E_1

$$E_1 = E_f V_f + E_m V_m$$

Major Poisson's ratio, ν_{12}

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$

Transverse elastic modulus, E_2

$$\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$$

In-plane shear modulus, G_{12}

$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}$$

END

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Chapter 3 Micromechanical Analysis of a Lamina
Elastic Moduli of Unidirectional Lamina
Halpin-Tsai Equations

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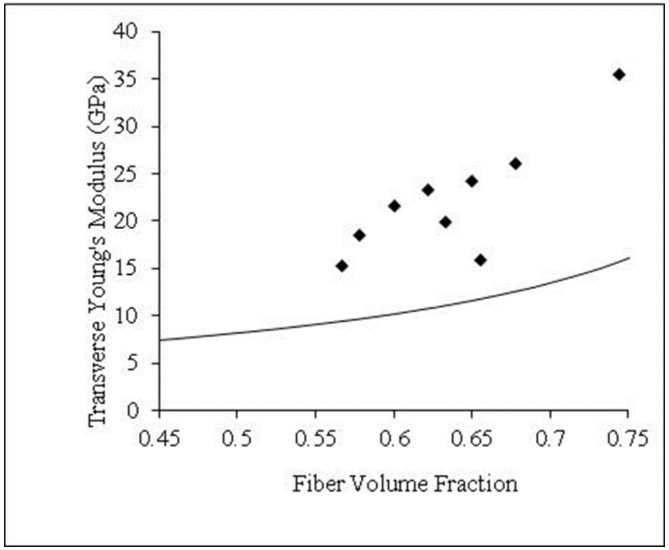
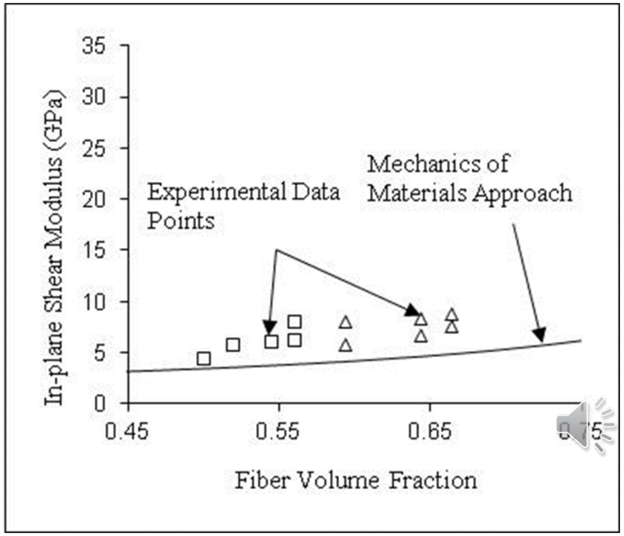
Halpin Tsai Equations

Same as MOM equations for E1 and ν_{12} but NOT for E2 and G12

$$E_1 = E_f V_f + E_m V_m$$

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$





Halpin Tsai Equations: Transverse Young's Modulus

$$\frac{E_2}{E_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f} \qquad \eta = \frac{(E_f / E_m) - 1}{(E_f / E_m) + \xi}$$

The term ξ is called the reinforcing factor and depends on the following

- Fiber Geometry
- Packing Geometry
- Loading Conditions

Example: For a fiber geometry of circular fibers in a packing geometry of a square array, $\xi = 2$

Does η have meaning?

$$\frac{E_2}{E_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f}$$

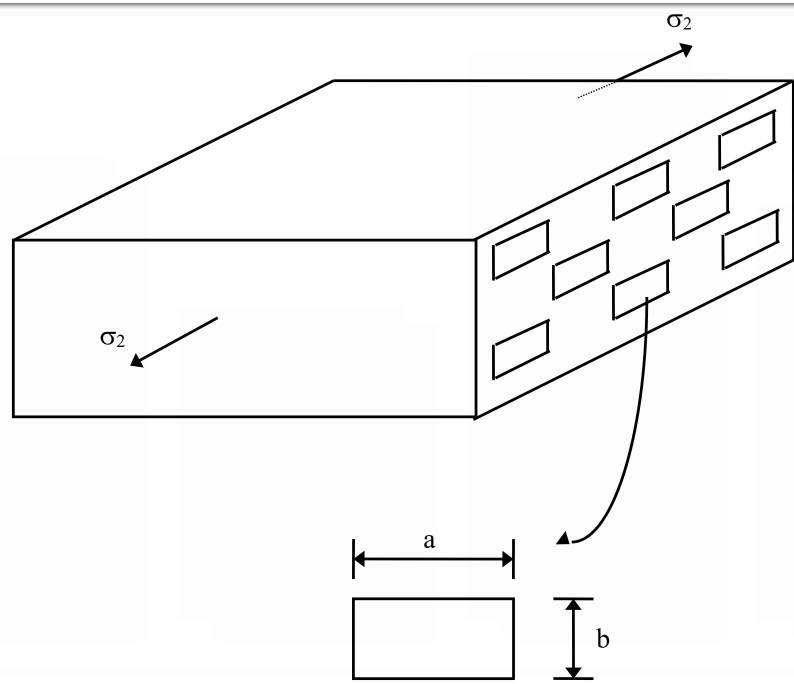
$$\eta = \frac{(E_f / E_m) - 1}{(E_f / E_m) + \xi}$$

$E_f/E_m = 1$ implies $\eta = 0$, (homogeneous medium)

$E_f/E_m \rightarrow \infty$ implies $\eta = 1$, (rigid inclusions)

$E_f/E_m \rightarrow 0$ implies $\eta = -\frac{1}{\xi}$ (voids)

What is ξ for another case?



The term ξ is depends on

- Fiber Geometry
- Packing Geometry
- Loading Conditions

For a rectangular fiber cross-section of length *a* and width *b* in a hexagonal array,
$\xi = 2(a/b)$,
where *a* is in the direction of loading.

Example: Transverse Young's Modulus

Find the transverse Young's modulus for a Glass/Epoxy lamina with a 70% fiber volume fraction. Use the properties for glass and epoxy from Tables 3.1 and 3.2, respectively. Use Halpin-Tsai equations for a circular fiber in a square array packing geometry.

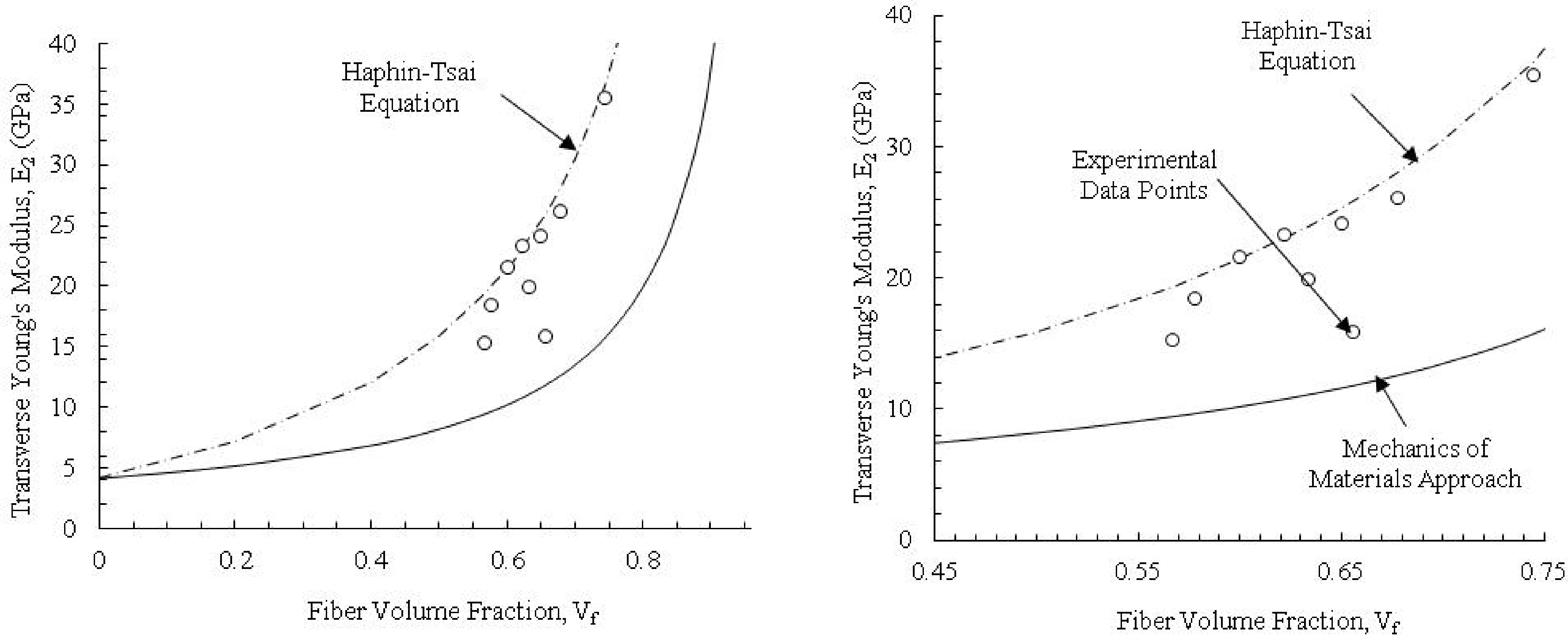
$$\xi = 2 \qquad E_f = 85\ GPa \qquad E_m = 3.4\ GPa$$

$$\eta = \frac{(E_f / E_m) - 1}{(E_f / E_m) + \xi} \qquad \eta = \frac{(85/3.4) - 1}{(85/3.4) + 2} = 0.8889$$

$$\frac{E_2}{E_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f} \qquad \frac{E_2}{3.4} = \frac{1 + 2(0.8889)(0.7)}{1 - (0.8889)(0.7)}$$

$$E_2 = 20.20\ GPa$$

Transverse Young's Modulus MOM vs HT



Theoretical values of transverse Young's modulus as a function of fiber volume fraction and comparison with experimental values for boron/epoxy unidirectional lamina (*Ef* = 414 GPa, v*f* = 0.2, *Em* = 4.14 GPa, v*m* = 0.35). Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract no. NAS1-8818, November 1970.)

In-plane Shear Modulus

$$\frac{G_{12}}{G_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f}$$

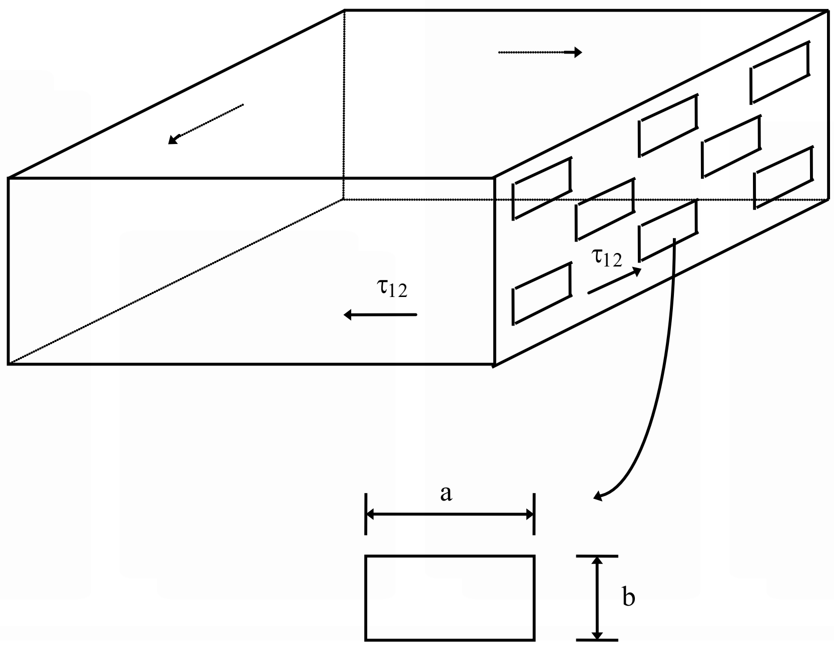
$$\eta = \frac{(G_f / G_m) - 1}{(G_f / G_m) + \xi}$$

The term ξ is called the reinforcing factor and depends on the following

- Fiber Geometry
- Packing Geometry
- Loading Conditions

Example: For a fiber geometry of circular fibers in a packing geometry of a square array, $\xi = 1$

What is ξ for another case?



The term ξ is depends on

- Fiber Geometry
- Packing Geometry
- Loading Conditions

For a rectangular fiber cross-section of length a and width b in a hexagonal array,
$\xi = \sqrt{3}\ln(\frac{a}{b})$,
where a is in the direction of loading

Example: In-Plane Shear Modulus

Using Halpin-Tsai equations, find the shear modulus of a Glass/Epoxy composite with a 70% fiber volume fraction. Use the properties of glass and epoxy from Tables 3.1 and 3.2, respectively. Assume the fibers are circular and are packed in a square array.

$$\xi = 1 \qquad G_f = 35.42\ GPa \qquad G_m = 1.308\ GPa$$

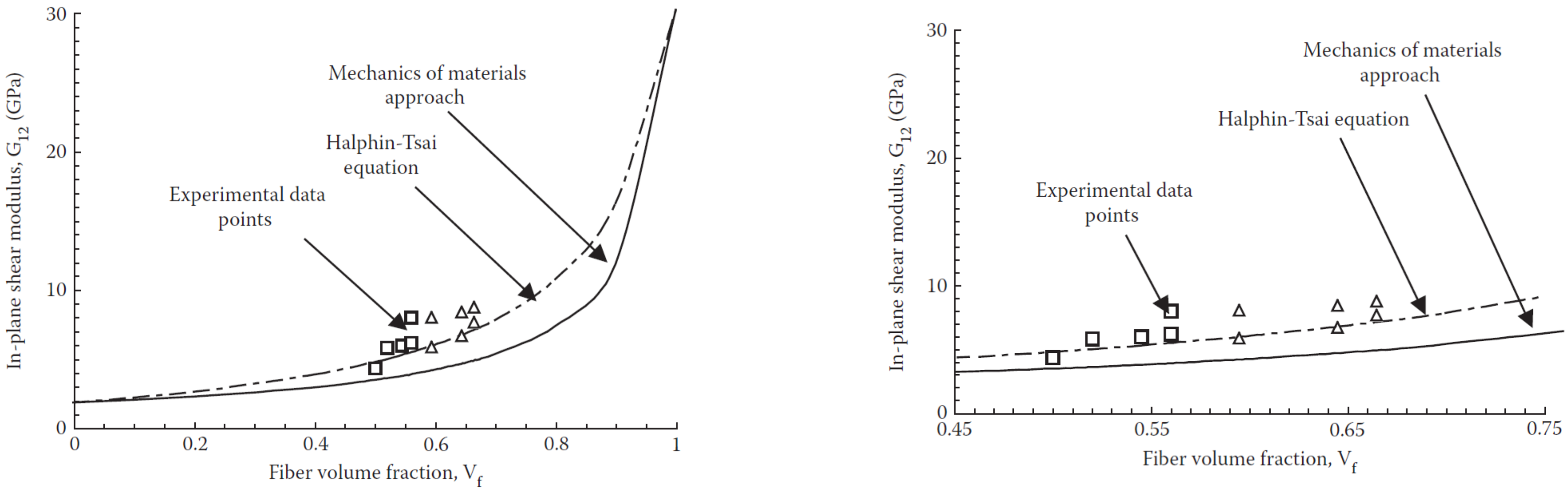
$$\eta = \frac{(35.42/1.308) - 1}{(35.42/1.308) + 1}$$

$$= 0.9288$$

$$\frac{G_{12}}{1.308} = \frac{1 + (1)(0.9288)(0.7)}{1 - (0.9288)(0.7)}$$

$$G_{12} = 6.169\ GPa$$

Shear Modulus MOM vs HT



Theoretical values of in-plane shear modulus as a function of fiber volume fraction compared with experimental values for unidirectional glass/epoxy lamina (Gf = 30.19 GPa, Gm = 1.83 GPa). Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract No. NAS1-8818, November 1970.)

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