# EML 4230 Introduction to Composite Materials

## Chapter 4 Macromechanical Analysis of a Laminate
## **Laminate Modulus**


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Courtesy of the Textbook
**Mechanics of Composite Materials by Kaw**

# In-Plane and Flexural Modulus of a Laminate

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$[N] = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} \qquad [\varepsilon^0] = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix}$$

$$\begin{bmatrix} \underline{N} \\ M \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B} \\ B & D \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ \hline \kappa \end{bmatrix}$$

$$[M] = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} \qquad [\kappa] = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

# In-Plane and Flexural Modulus of a Laminate

$$\begin{bmatrix} \varepsilon^0 \\ \hline \kappa \end{bmatrix} = \begin{bmatrix} A^* & B^* \\ \hline C^* & D^* \end{bmatrix} \begin{bmatrix} N \\ \hline M \end{bmatrix}$$

$$\begin{bmatrix} A^* & B^* \\ \hline C^* & D^* \end{bmatrix} = \begin{bmatrix} A & B \\ \hline B & D \end{bmatrix}^{-1}$$

$$[C^*] = [B^*]^T$$

The [A*], [B*], and [D*] matrices are called the extensional compliance matrix, coupling compliance matrix, and bending compliance matrix respectively.

# In-Plane Engineering Constants of a Laminate

For a symmetric laminate:

$$[B] = 0,$$

$$A_{ij} = \sum_{k=1}^{n} [(\overline{Q}_{ij})]_k \ (h_k - h_{k-1}), \ i = 1, 2, 6; \ j = 1, 2, 6$$

$$[A^*] = [A]^{-1},$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} [(\overline{Q}_{ij})]_k \ (h_k^2 - h_{k-1}^2), \ i = 1, 2, 6; \ j = 1, 2, 6$$

$$[D^*] = [D]^{-1}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} [(\overline{Q}_{ij})]_k \ (h_k^3 - h_{k-1}^3) \ i = 1, 2, 6; \ j = 1, 2, 6$$

# In-Plane Engineering Constants of a Laminate

$$\begin{bmatrix} \varepsilon^0 \\ \hline \kappa \end{bmatrix} = \begin{bmatrix} A^* & B^* \\ \hline C^* & D^* \end{bmatrix} \begin{bmatrix} N \\ \hline M \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}$$

# In-Plane Engineering Constants of a Laminate

$$\text{Effective in} - \text{plane longitudinal modulus } E_x$$

$$N_x \neq 0, N_y = 0, N_{xy} = 0$$

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} N_x \\ 0 \\ 0 \end{bmatrix}$$

$$\varepsilon_x^0 = A_{11}^* N_x$$

$$E_x \equiv \frac{\sigma_x}{\varepsilon_x^0} = \frac{N_x / h}{A_{11}^* N_x} = \frac{1}{h A_{11}^*}$$

# In-Plane Engineering Constants of a Laminate

$$\text{Effective in}-\text{plane transverse modulus } \mathrm{E}_{\mathrm{y}}$$

$$N_x = 0, N_y \neq 0, N_{xy} = 0$$

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} 0 \\ N_y \\ 0 \end{bmatrix}$$

$$\varepsilon_y^0 = A_{22}^* N_y$$

$$E_y \equiv \frac{\sigma_y}{\varepsilon_y^0} = \frac{N_y / h}{A_{22}^* N_y} = \frac{1}{h A_{22}^*}$$

# In-Plane Engineering Constants of a Laminate

$$\text{Effective in – plane shear modulus } G_{xy}$$

$$N_x = 0, N_y = 0, N_{xy} \neq 0$$

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ N_{xy} \end{bmatrix}$$

$$\gamma_{xy}^0 = A_{66}^* N_{xy}$$

$$G_{xy} \equiv \frac{\tau_{xy}}{\gamma_{xy}^0} = \frac{N_{xy} / h}{A_{66}^* N_{xy}} = \frac{1}{h A_{66}^*}$$

# In-Plane Engineering Constants of a Laminate

$$\text{Effective in – plane Poisson's ratio } \nu_{xy}$$

$$N_x \neq 0, N_y = 0, N_{xy} = 0$$

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} N_x \\ 0 \\ 0 \end{bmatrix}$$

$$\varepsilon_y^0 = A_{12}^* N_x$$

$$\varepsilon_x^0 = A_{11}^* N_x$$

$$\nu_{xy} \equiv -\frac{\varepsilon_y^0}{\varepsilon_x^0} = -\frac{A_{12}^* N_x}{A_{11}^* N_x} = -\frac{A_{12}^*}{A_{11}^*}$$

# In-Plane Engineering Constants of a Laminate

$$\text{Effective in} - \text{plane Poisson's ratio } \nu_{yx}$$

$$N_x = 0, N_y \neq 0, N_{xy} = 0$$

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} 0 \\ N_y \\ 0 \end{bmatrix}$$

$$\varepsilon_x^0 = A_{12}^* N_y$$

$$\varepsilon_y^0 = A_{22}^* N_y$$

$$\nu_{yx} \equiv -\frac{\varepsilon_x^0}{\varepsilon_y^0}$$

$$= -\frac{A_{12}^*}{A_{22}^*}\frac{N_y}{N_y}$$

$$= -\frac{A_{12}^*}{A_{22}^*}$$

# In-Plane Engineering Constants of a Laminate

$$\frac{\nu_{xy}}{E_x} = \left( - \frac{A_{12}^*}{A_{11}^*} \right) h \, A_{11}^*$$

$$= - A_{12}^* h$$

$$\frac{\nu_{yx}}{E_y} = \left( - \frac{A_{12}^*}{A_{22}^*} \right) h \, A_{22}^*$$

$$= - A_{12}^* h$$

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}$$

# Flexural Engineering Constants of a Laminate

For a symmetric laminate:

[B] = 0

$$\left[\frac{\varepsilon^0}{\kappa}\right] = \left[\begin{array}{c|c} A^* & B^* \\ \hline C^* & D^* \end{array}\right] \left[\frac{N}{M}\right]$$

$$[C^*] = [B^*]^T$$

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{12}^* & D_{22}^* & D_{26}^* \\ D_{16}^* & D_{26}^* & D_{66}^* \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}$$

# Flexural Engineering Constants of a Laminate

Effective flexural longitudinal modulus $E_x^f$

$$M_x \neq 0, M_y = 0, M_{xy} = 0$$

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{12}^* & D_{22}^* & D_{26}^* \\ D_{16}^* & D_{26}^* & D_{66}^* \end{bmatrix} \begin{bmatrix} M_x \\ 0 \\ 0 \end{bmatrix}$$

$$\kappa_x = D_{11}^* M_x$$

$$E_x^f \equiv \frac{12 M_x}{\kappa_x h^3} = \frac{12}{h^3 D_{11}^*}$$

# Flexural Engineering Constants of a Laminate

Other flexural elastic moduli :

$$E_y^f = \frac{12}{h^3 D_{22}^*}$$

$$G_{xy}^f = \frac{12}{h^3 D_{66}^*}$$

$$\nu_{xy}^f = -\frac{D_{12}^*}{D_{11}^*}$$

$$\nu_{yx}^f = -\frac{D_{12}^*}{D_{22}^*}$$

$$\frac{\nu_{xy}^f}{E_x^f} = \frac{\nu_{yx}^f}{E_y^f}$$

# END