LU Decomposition Method

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Transforming Numerical Methods Education for STEM Undergraduates

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Questions to have been asked and to ask?

- What is LU decomposition method?
- How do we decompose a coefficient matrix to LU?
- LU decomposition method looks so much like Naïve Gauss elimination, what gives?
- How do we use LU decomposition method to find inverse of a matrix?

Reviewing the LU Decomposition example

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$
$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \text{ gives } \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix} \text{ gives } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

LU Decomposition Method

[A][X] = [C]

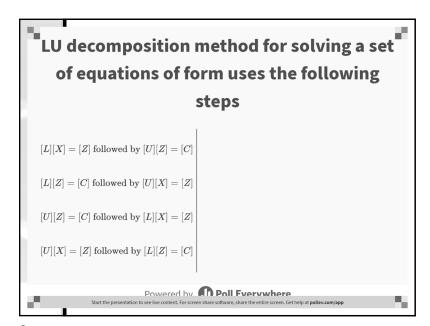
- 1.Decompose [A] into [L] and [U]
- 2.Solve [L][Z] = [C] for [Z] by using forward substitution
- 3. Solve [U][X] = [Z] for [X] by using back substitution

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To solve a set of equations [A][X] = [C] by LU decomposition, the first set of equations to be solved are [L][Z] = [C]. The vector [Z] is ______. (Check all that apply)

Same as $[L]^{-1}[C]$ Same as the solution vector [X]Same as the vector [C]



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Every A matrix that has inverse can be decomposed to LU or PLU

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$A = PLU$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -8 & 8 & 1 \\ 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

P is a so-called permutation matrix

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Is LU Decomposition better than Gaussian Elimination?

THE END

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Naïve Gaussian Elimination

[A][X] = [C]

1.Conduct forward elimination to get [U][X] = [Z]

2. Conduct back substitution to solve[U][X] = [Z] for [X]

LU Decomposition Method

3. Solve [U][X] = [Z] for [X] by using back substitution

[A][X] = [C]

1.Decompose [A] into [L] and [U]

2.Solve [L][Z] = [C] for [Z] by using forward substitution

Solve [A][X] = [B] $T = \operatorname{clock}$ cycle time $n \times n = \operatorname{size}$ of the matrix $CT = \operatorname{computational}$ time

Naive Gauss Elimination Method

Forward Elimination $CT|_{FE} = T\left(\frac{8n^3}{3} + 8n^2 - \frac{32n}{3}\right)$ Back Substitution $CT|_{BS} = T(4n^2 + 12n)$ Forward Substitution $CT|_{BS} = T(4n^2 - 4n)$ Back Substitution $CT|_{BS} = T(4n^2 + 12n)$

Is LU Decomposition Method better than

Gaussian Elimination Method?

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Is LU Decomposition better than Gaussian Elimination?

To solve [A][X] = [B]

Time taken by methods

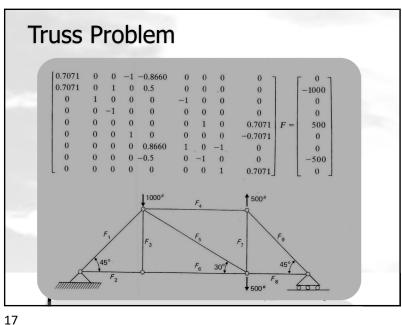
Gaussian Elimination	LU Decomposition	
$T\left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3}\right)$	$T\left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3}\right)$	

T =clock cycle time and $n \times n =$ size of the matrix

So both methods are equally efficient.



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Finding the inverse of a square matrix

[B] is inverse of a square matrix [A] if

$$[A][B] = [I]$$
 OR $[B][A] = [I]$

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Two ways to show what inverse means?

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{21} & b_{22} & b_{23} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finding the inverse of a square matrix

How can LU Decomposition be used to find the inverse?

$$[A][B] = [I]$$

First Column of [B]

Second Column of [B]

Last Column of [B]

Example: Inverse of a Matrix

Find the inverse of a square matrix [A]

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Using the decomposition procedure, the [L] and [U] matrices are found to be

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Solving for each column of [B] requires two steps

- 1) Solve [L][Z] = [C] for [Z]
- 2) Solve [U][X] = [Z] for [X]

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Finding 1st column of inverse: UX=Z $Step 2: [U][X] = [Z] \rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$ $\begin{vmatrix} b_{31} = \frac{3.2}{0.7} \\ = 4.571 \\ b_{21} = \frac{-2.56 + 1.560b_{31}}{-4.8} \\ = \frac{-2.56 + 1.560(4.571)}{-4.8} \\ = \frac{-0.9524}{0.7b_{31}} = 3.2 \end{vmatrix}$ $\begin{vmatrix} b_{11} = \frac{b_{11}}{b_{21}} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ b_{11} = \frac{1 - 5b_{21} - b_{31}}{25} \\ = 0.04762 \end{vmatrix}$

Finding 1st column of inverse: LZ=C
$$\begin{bmatrix}
25 & 5 & 1 \\
64 & 8 & 1 \\
144 & 12 & 1
\end{bmatrix}
\begin{bmatrix}
b_{11} \\
b_{21} \\
b_{31}
\end{bmatrix} = \begin{bmatrix}
1 \\ 0 \\
0
\end{bmatrix}$$

$$Step 1: [L][Z] = [C] \rightarrow \begin{bmatrix}
1 & 0 & 0 \\
2.56 & 1 & 0 \\
5.76 & 3.5 & 1
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}$$

$$\begin{bmatrix}
z_1 = 1 \\
z_2 = 0 - 2.56z_1 \\
= 0 - 2.56(1) \\
= -2.56 \\
z_3 = 0 - 5.76z_1 - 3.5z_2 \\
= 0 - 5.76(1) - 3.5(-2.56) \\
= 3.2
\end{bmatrix}$$

$$[Z] = \begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix} = \begin{bmatrix}
1 \\
-2.56 \\
3.2
\end{bmatrix}$$

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Inverse of a Matrix $ \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $						
First Column	Second Column	Third Column				
$ \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} $	$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$				
$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$	$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$	$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$				

Example: Inverse of a Matrix

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix} \qquad \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix} \qquad \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

To check your work, check the following

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

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To find inverse of [A]

 $T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$

Time taken by LU Decomposition $T\left(\frac{32n^3}{3} + 12n^2 - \frac{20n}{3}\right)$

Table 1 Comparing computational times of finding inverse of a matrix using LU decomposition and Naive Gauss elimination.

n	10	100	1000	10000
$\frac{CT_{\text{inverse GE}}}{CT_{\text{inverse LU}}}$	3.288	25.84	250.8	2501

For large
$$n$$
, $\frac{CT_{\text{inverse GE}}}{CT_{\text{inverse LU}}} \approx \frac{n}{4}$

To find inverse of [A]

Time taken by Gaussian Elimination

$$= n \times T \left(\frac{8n^3}{3} + 8n^2 - \frac{32n}{3} \right)$$
$$+ n \times T (4n^2 + 12n)$$

 $= n(CT|_{FE} + CT|_{RS})$

$$= T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

Time taken by LU Decomposition

$$= CT|_{DE} + n \times CT|_{FS} + n \times CT|_{BS}$$

$$= T\left(\frac{8n^3}{3} + 4n^2 - \frac{20n}{3}\right) + n \times T(4n^2 - 4n) + n \times T(4n^2 + 12n)$$

$$= T \left(\frac{32n^3}{3} + 12n^2 - \frac{20n}{3} \right)$$

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Time Taken by Back Substitution

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a'_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b_3 \\ \vdots \\ b'_{n} \end{bmatrix}$$

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

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$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^n a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \dots, 1$$

n = length(B);X(n)=B(n)/A(n,n)

for i=n-1:-1:1

X(i) = B(i);

for j=(i+1):1:nX(i) = X(i) - A(i, j)*X(j);

nd

X(i) = X(i)/A(i, i);

Back Substitution CT

 $CT|_{BS} = T(4n^2 + 12n)$

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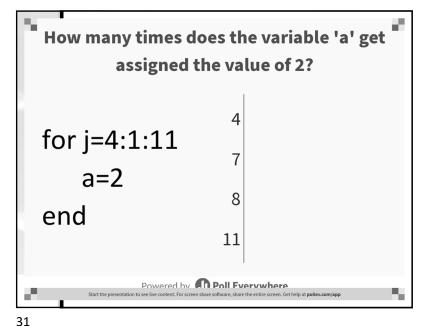
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Time Taken by Back Substitution n = length(B); $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a_{33}^* & \cdots & a_{3n}^* \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$ X(n)=B(n)/A(n,n)for i=n-1:-1:1 X(i) = B(i);for j=(i+1):1:nX(i) = X(i) - A(i, j) * X(j);end X(i) = X(i)/A(i, i);**Back Substitution CT** $CT|_{BS} = T(4n^2 + 12n)$

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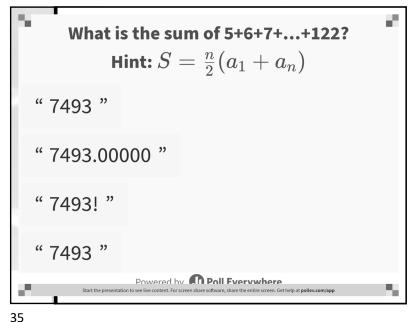


How many times does the variable 'a' get assigned the value of 2? n-i for j=i+1:1:n n-i-1 a=2 end n i+1 Powered by Poll Fverywhere

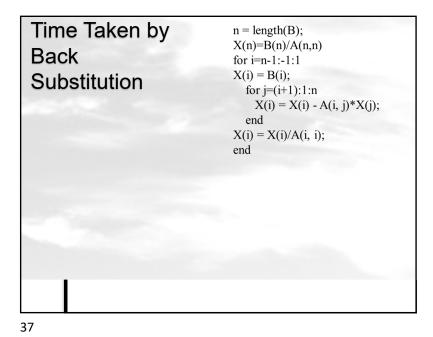
How many times does the variable 'a' get assigned the value of 2? for i=3: -1: 1 for j=i+1: 1: 4 a=2 end end 12 Powered by Poll Fverywhere

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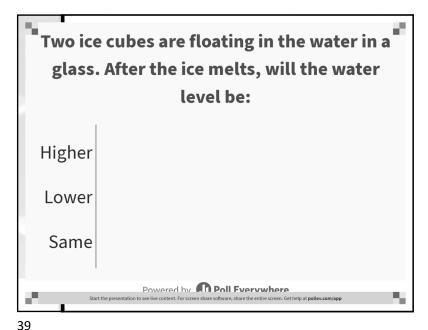
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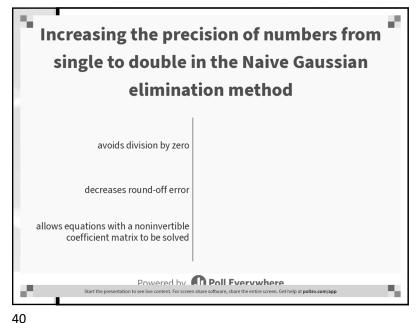


```
Time Taken by Back Substitution
                                      n = length(B);
                                      X(n)=B(n)/A(n,n)
                                      for i=n-1:-1:1
                                      X(i) = B(i);
                                         for j=(i+1):1:n
                                          X(i) = X(i) - A(i, j) * X(j);
x_n = \frac{b_n^{(n-1)}}{a_{--}^{(n-1)}}
                                         end
                                      X(i) = X(i)/A(i, i);
                                       Back Substitution CT
                                        CT|_{BS} = T(4n^2 + 12n)
```



THE END





Someone is asking you to use LU

decomposition method to solve m sets of (n

equations, n unknowns) of the form [A][X]=

[C]. If the coefficient matrix [A] stays the

same in all sets, how many times does one

need to decompose the matrix [A]?

There are two parts in Naive Gaussian

Elimination - forward elimination and back
substitution. If we are solving n equations
for n unknowns, how many steps are in the
forward elimination part?

n
n-1
n+1

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Naive Gaussian elimination has these inherent numerical methods errors (Check all that apply)

Round off errors

Truncation errors

Apple Foreverbere

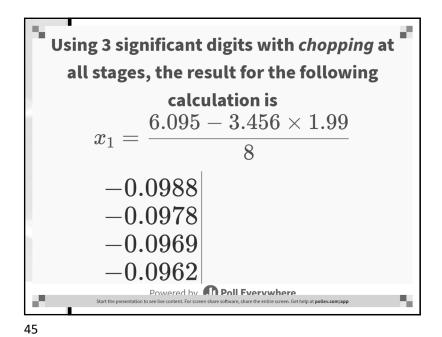
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LU decomposition method is computationally more efficient than Naive Gauss elimination for (Check all that apply)

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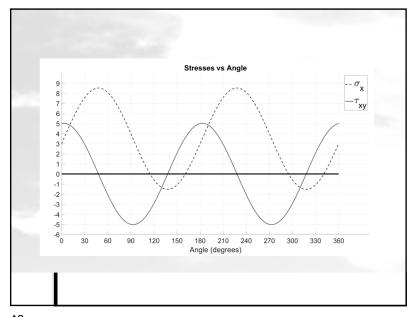


Division by zero during the forward elimination part in Naïve Gaussian elimination for [A][X]=[C] implies the coefficient matrix [A]

is invertible
is not invertible
cannot be determined to be invertible or not

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