EML3041 Computational Methods
Spring 2023

Answer each question in sequence on a fresh sheet of paper. Solve the problem as if you were submitting them for a test. Identify each part separately if a question has parts.

1) The one-point Gauss quadrature rule of integration is derived by starting with

$$\int_a^b f(x)dx \approx c_1 f(x_1), \text{ where } a \le x_1 \le b.$$

The values of c_1 and x_1 are found by assuming that the above formula is exact for the polynomial integrand of the form $a_0 + a_1 x$ polynomial. We obtain the one-point Gauss quadrature rule as

$$\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right) \qquad (1)$$

a) What would be the highest order polynomial integrand for which equation (1) be exact?
b) What is the exact value of the integral $\int_2^7 (3x^2 + 5x)dx$?
c) Estimate the value of the integral $\int_2^7 (3x^2 + 5x)dx$ by using the one-point Gauss quadrature rule.
d) Is the answer in part (c) exact? If not, what is the true error?
e) Did you expect it to give the exact value for part (c), and why or why not?

An engineer named Adele does not like Gauss for reasons known to us all. She develops an approximate formula for integration as

$$\int_a^b f(x)dx \approx c_1 f(x_1), \text{ where } a \le x_1 \le b$$

The values of c_1 and x_1 were found by Adele by assuming that the above formula is exact for the polynomials of the form $a_0 x + a_1 x^2$. She calls the rule – one-point Adele quadrature rule.
f) Find the values of c_1 and x_1.
g) Verify if the one-point Adele quadrature formula gives the exact value for the integral $\int_2^7 (3x^2 + 5x)dx$.
h) Verify if the one-point Adele quadrature formula gives the exact value for the integral of $\int_2^7 2dx$.
i) Based on what you have seen so far, was this a good idea for Adele to have chosen $a_0 x + a_1 x^2$ polynomial for deriving her one-point rule? Why or why not?

Answer
a) Answer not given intentionally
b) 447.50
c) 416.25
d) Answer to the first part not given intentionally; True error=31.25
e) Answer not given intentionally
f) $c_1 = \frac{3(b-a)(b+a)^2}{4(b^2+a^2+ab)}$; $x_1 = \frac{2(b^2+a^2+ab)}{3(b+a)}$
g) 447.50 (from the formula); 447.50 (from exact)
h) 9.067 (from the formula); 10 (from exact)

i) Not given intentionally

2. a) Show the full derivation of finding the general formula for regressing n data points $(x_1, y_1), (x_2, y_2), \dots \dots, (x_n, y_n)$ to $y = kx^2$.
b) Did you forget to do the second derivative test and establish that the answer in part (a) in fact corresponds to a local minimum of the sum of square of residuals? If you did not, a good time to conduct the test is now because you should have done this in part (a). Is this local minimum also the absolute minimum? If so, how do you reason that out?

c) The following force vs. displacement data is given for a nonlinear spring

Displacement, x (m)	10	15	20
Force, F (m)	100	200	400

Estimate the value of k if $F = kx^2$ is the regression model.

Answer: (a) $k = \frac{\sum_{i=1}^{n} y_i x_i^2}{\sum_{i=1}^{n} x_i^4}$ (b) Not given intentionally (c) $0.9745\ N/m^2$

3) What is the value of the integral $\int_0^{2.4} f(x)dx$ if

$f(x) = 2.4x, 0 \le x \le 0.22$ and

$= 3.2x^2, 0.22 < x \le 2.4,$

using the composite trapezoidal rule with two segments?

Answer: 16.588

4) The upward velocity of a rocket is given by

$$v(t) = 200\ln(t+1) - 10t,\ t > 0$$

where t is given in seconds and v is given in m/s.

a) Use 2-point Gauss quadrature rule to calculate the displacement of the rocket from $t = 0$ to $t = 5$s.
b) What is the true value of the displacement of the rocket from $t = 0$ to $t = 5$s?
c) What is the true error in part (a)?
d) What is the relative true error in part (a)?
e) What is the absolute relative true error in percentage for part (a).

Answer:

$a)$ $1034.6\ m$

$b)$ $1025.1\ m$

$c)$ $-9.4458\ m$

$d)$ -0.0092144

$e)$ 0.92144%