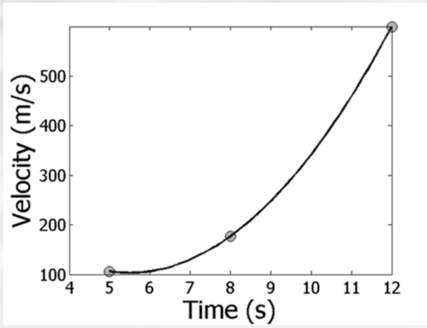
Interpolation - Introduction

Reading Between the Lines



1

WHAT IS INTERPOLATION ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.

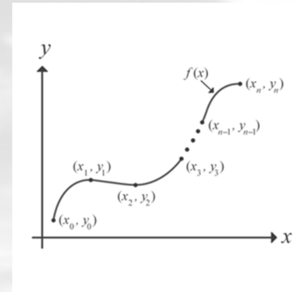


Figure Interpolation of discrete data.

http://nm.mathforcollege.com

2

Spline Method of Interpolation

http://nm.MathForCollege.com

3

Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

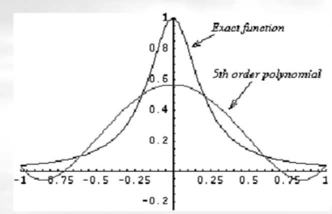


Figure : 5th order polynomial vs. exact function

4

http://nm.MathForCollege.com

4

Why Splines ?

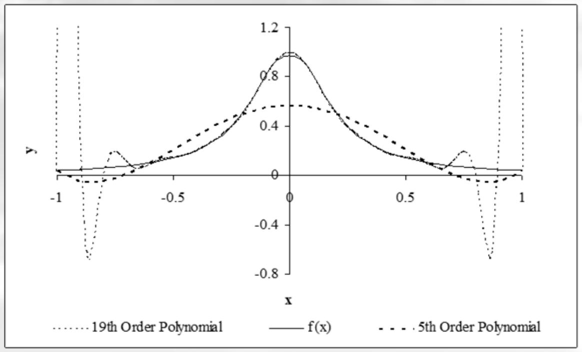


Figure : Higher order polynomial interpolation is a bad idea

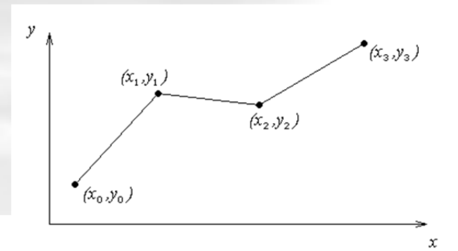
5 http://nm.MathForCollege.com

5

Linear Spline Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $(y_i = f(x_i))$

Figure : Linear splines



6 http://nm.MathForCollege.com

6

Linear Spline Interpolation (contd)

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2$$

.

.

.

$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between x_{i-1} and x_i.

7 http://nm.MathForCollege.com

7

Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using linear splines.

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

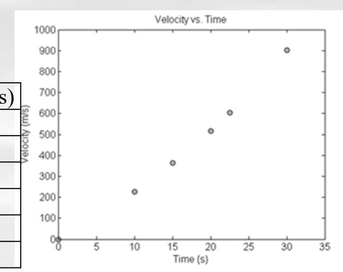


Figure. Velocity vs. time data for the rocket example

8 http://nm.MathForCollege.com

8

Linear Spline Interpolation

$t_0 = 15, \qquad v(t_0) = 362.78$

$t_1 = 20, \qquad v(t_1) = 517.35$

$$v(t) = v(t_0) + \frac{v(t_1) - v(t_0)}{t_1 - t_0}(t - t_0)$$

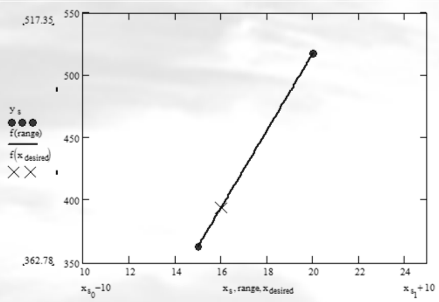
$$= 362.78 + \frac{517.35 - 362.78}{20 - 15}(t - 15)$$

$$v(t) = 362.78 + 30.913(t - 15)$$

At $t = 16$,

$$v(16) = 362.78 + 30.913(16 - 15)$$

$$= 393.7 \text{ m/s}$$



9

Quadratic Spline Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$f(x) = a_1x^2 + b_1x + c_1, \qquad x_0 \le x \le x_1$$

$$= a_2x^2 + b_2x + c_2, \qquad x_1 \le x \le x_2$$

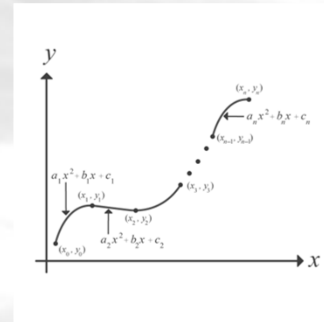
.

.

.

$$= a_nx^2 + b_nx + c_n, \qquad x_{n-1} \le x \le x_n$$

Find $a_i, b_i, c_i, i = 1, 2, \ldots, n$



10

Quadratic Spline Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$a_1x_0^2 + b_1x_0 + c_1 = f(x_0)$$

$$a_1x_1^2 + b_1x_1 + c_1 = f(x_1)$$

.

.

$$a_ix_{i-1}^2 + b_ix_{i-1} + c_i = f(x_{i-1})$$

$$a_ix_i^2 + b_ix_i + c_i = f(x_i)$$

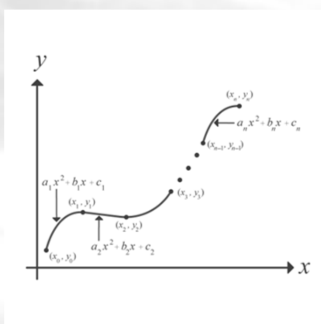
.

.

$$a_nx_{n-1}^2 + b_nx_{n-1} + c_n = f(x_{n-1})$$

$$a_nx_n^2 + b_nx_n + c_n = f(x_n)$$

This condition gives 2n equations



11

Quadratic Spline Interpolation (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$a_1x^2 + b_1x + c_1$ is $2a_1x + b_1$

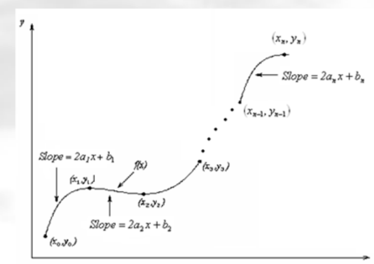
The derivative of the second spline

$a_2x^2 + b_2x + c_2$ is $2a_2x + b_2$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



12

Quadratic Spline Interpolation (contd)

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

.

.

.

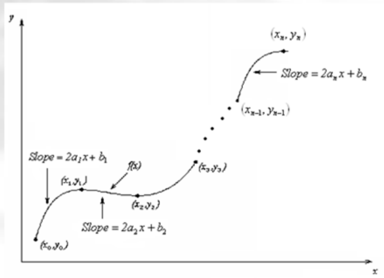
$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

.

.

.

$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



We have (n-1) such equations. The total number of equations is $(2n) + (n-1) = (3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

13 http://nm.MathForCollege.com

13

Quadratic Spline Interpolation (contd)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

$$f(x) = a_1x^2 + b_1x + c_1, \quad x_0 \le x \le x_1$$

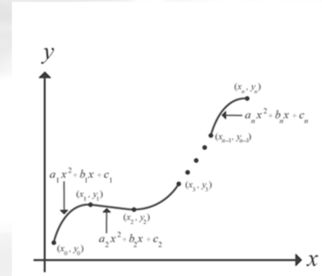
$$= a_2x^2 + b_2x + c_2, \quad x_1 \le x \le x_2$$

.

.

.

$$= a_nx^2 + b_nx + c_n, \quad x_{n-1} \le x \le x_n$$



14 http://nm.MathForCollege.com

14

Quadratic Spline Example

The upward velocity of a rocket is given as a function of time. Using quadratic splines

a) Find the velocity at t=16 seconds

b) Find the acceleration at t=16 seconds

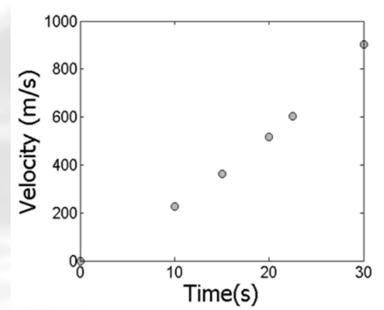
c) Find the distance covered between t=11 and t=16 seconds

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

15

Data and Plot

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



16

Solution

$$v(t) = a_1t^2 + b_1t + c_1,\ 0 \le t \le 10$$

$$= a_2t^2 + b_2t + c_2,\ 10 \le t \le 15$$

$$= a_3t^2 + b_3t + c_3,\ 15 \le t \le 20$$

$$= a_4t^2 + b_4t + c_4,\ 20 \le t \le 22.5$$

$$= a_5t^2 + b_5t + c_5,\ \ 22.5 \le t \le 30$$

Let us set up the equations

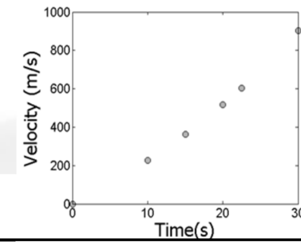
17

Each Spline Goes Through Two Consecutive Data Points

$$v(t) = a_1t^2 + b_1t + c_1,\ 0 \le t \le 10$$

$$a_1(0)^2 + b_1(0) + c_1 = 0$$

$$a_1(10)^2 + b_1(10) + c_1 = 227.04$$



18

Each Spline Goes Through Two Consecutive Data Points

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$a_2(10)^2 + b_2(10) + c_2 = 227.04$$

$$a_2(15)^2 + b_2(15) + c_2 = 362.78$$

$$a_3(15)^2 + b_3(15) + c_3 = 362.78$$

$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$

$$a_4(20)^2 + b_4(20) + c_4 = 517.35$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97$$

$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

19

Derivatives are Continuous at Interior Data Points

$$v(t) = a_1t^2 + b_1t + c_1,\ 0 \le t \le 10$$

$$= a_2t^2 + b_2t + c_2,\ 10 \le t \le 15$$

$$\frac{d}{dt}(a_1t^2 + b_1t + c_1)\Big|_{t=10} = \frac{d}{dt}(a_2t^2 + b_2t + c_2)\Big|_{t=10}$$

$$(2a_1t + b_1)\Big|_{t=10} = (2a_2t + b_2)\Big|_{t=10}$$

$$2a_1(10) + b_1 = 2a_2(10) + b_2$$

$$20a_1 + b_1 - 20a_2 - b_2 = 0$$

20

Derivatives are continuous at Interior Data Points

At t=10
$$2a_1(10) + b_1 - 2a_2(10) - b_2 = 0$$

At t=15
$$2a_2(15) + b_2 - 2a_3(15) - b_3 = 0$$

At t=20
$$2a_3(20) + b_3 - 2a_4(20) - b_4 = 0$$

At t=22.5
$$2a_4(22.5) + b_4 - 2a_5(22.5) - b_5 = 0$$

21

Last Equation

$$a_5 = 0$$

22

Final Set of Equations

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\ 20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

23

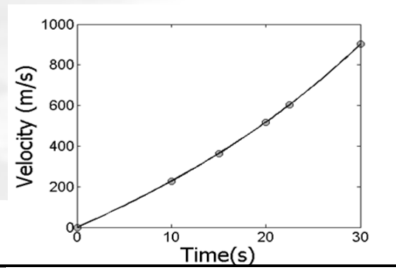
Coefficients of Spline

i	a_i	b_i	c_i
1	-0.15667	24.271	0
2	1.2021	-2.9053	135.88
3	-0.44893	46.627	-235.61
4	2.2315	-60.589	836.55
5	0	39.827	-293.13

24

Final Solution

$$\begin{aligned} v(t) &= -0.15667t^2 + 24.271t, \quad 0 \le t \le 10 \\ &= 1.2021t^2 - 2.9053t + 135.88, \quad 10 \le t \le 15 \\ &= -0.44893t^2 + 46.627t - 235.61, \quad 15 \le t \le 20 \\ &= 2.2315t^2 - 60.589t + 836.55, \quad 20 \le t \le 22.5 \\ &= 39.827t - 293.13, \quad 22.5 \le t \le 30 \end{aligned}$$



25

Velocity at a Particular Point

a) Velocity at t=16

$$\begin{aligned} v(t) &= -0.15667t^2 + 24.271t, \quad 0 \le t \le 10 \\ &= 1.2021t^2 - 2.9053t + 135.88, \quad 10 \le t \le 15 \\ &= -0.44893t^2 + 46.627t - 235.61, \quad 15 \le t \le 20 \\ &= 2.2315t^2 - 60.589t + 836.55, \quad 20 \le t \le 22.5 \\ &= 39.827t - 293.13, \quad 22.5 \le t \le 30 \end{aligned}$$

$$\begin{aligned} v(16) &= -0.44893(16)^2 + 46.627(16) - 235.61 \\ &= 395.50 \text{ m/s} \end{aligned}$$

26

Acceleration from Velocity Profile

b) Acceleration at t=16

$$\begin{aligned} v(t) &= -0.15667t^2 + 24.271t, && 0 \le t \le 10 \\ &= 1.2021t^2 - 2.9053t + 135.88, && 10 \le t \le 15 \\ &= -0.44893t^2 + 46.627t - 235.61, && 15 \le t \le 20 \\ &= 2.2315t^2 - 60.589t + 836.55, && 20 \le t \le 22.5 \\ &= 39.827t - 293.13, && 22.5 \le t \le 30 \end{aligned}$$

$$a(16) = \frac{d}{dt} v(t)\Big|_{t=16}$$

27

Acceleration from Velocity Profile

The quadratic spline valid at t=16 is given by

$$v(t) = -0.44893t^2 + 46.627t - 235.61, \quad 15 \le t \le 20$$

$$\begin{aligned} a(t) &= \frac{d}{dt}(-0.44893t^2 + 46.627t - 235.61) \\ &= -0.89786t + 46.627, \quad 15 \le t \le 20 \end{aligned}$$

$$\begin{aligned} a(16) &= -0.89786(16) + 46.627 \\ &= 32.261 \text{ m/s}^2 \end{aligned}$$

28

Distance from Velocity Profile

c) Find the distance covered by the rocket from t=11s to t=16s.

$$
\begin{aligned}
v(t) &= -0.15667t^2 + 24.271t, && 0 \le t \le 10 \\
&= 1.2021t^2 - 2.9053t + 135.88, && 10 \le t \le 15 \\
&= -0.44893t^2 + 46.627t - 235.61, && 15 \le t \le 20 \\
&= 2.2315t^2 - 60.589t + 836.55, && 20 \le t \le 22.5 \\
&= 39.827t - 293.13, && 22.5 \le t \le 30
\end{aligned}
$$

$$S(16) - S(11) = \int_{11}^{16} v(t)dt$$

29

Distance from Velocity Profile

$$
\begin{aligned}
v(t) &= 1.2021t^2 - 2.9053t + 135.88, \quad 10 \le t \le 15 \\
&= -0.44893t^2 + 46.627t - 235.61, \quad 15 \le t \le 20
\end{aligned}
$$

$$S(16) - S(11) = \int_{11}^{16} v(t)dt = \int_{11}^{15} v(t)dt + \int_{15}^{16} v(t)dt$$

$$= \int_{11}^{15} (1.2021t^2 - 2.9053t + 135.88)dt$$

$$+ \int_{15}^{16} (-0.44893t^2 + 46.627t - 235.61)dt$$

$$= 1590.7 \text{ m}$$

30

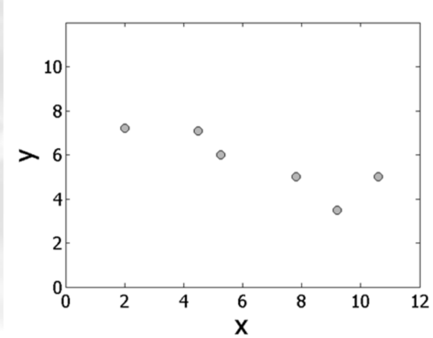
END

http://nm.MathForCollege.com

31

Points for Robot Path

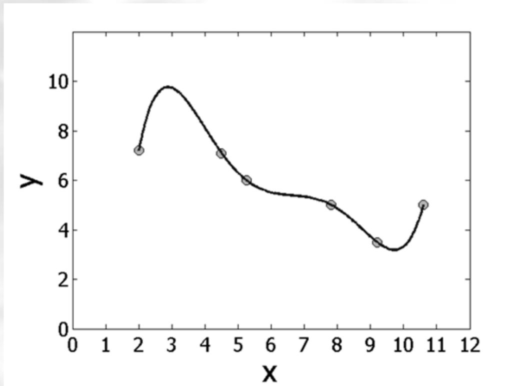
x	y
2.00	7.2
4.5	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0



Find the shortest but smooth path through consecutive data points

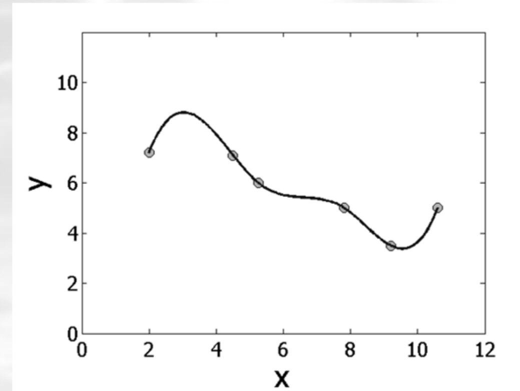
32

Polynomial Interpolant Path



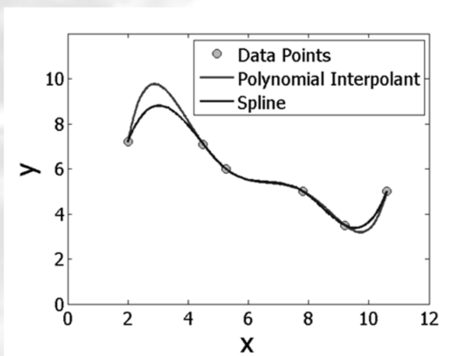
33

Spline Interpolant Path



34

Compare Spline & Polynomial Interpolant Path



Length of path

Polynomial Interpolant=14.9

Spline Interpolant =12.9

35

MathForCollege.com

Open Education Resources

36