Minimum of a twice differentiable continuous function

In regression, we are asked to minimize a differentiable, continuous function of one or two variables. In this primer, we will cover the basics of finding the minimum of a continuous function that is twice differentiable with domain D.

<u>Absolute Minimum Value</u>: Given a function $f(x)$ with domain D, then $f(c)$ is the absolute minimum on D if and only if $f(x) \geq f(c)$ for all x in D.

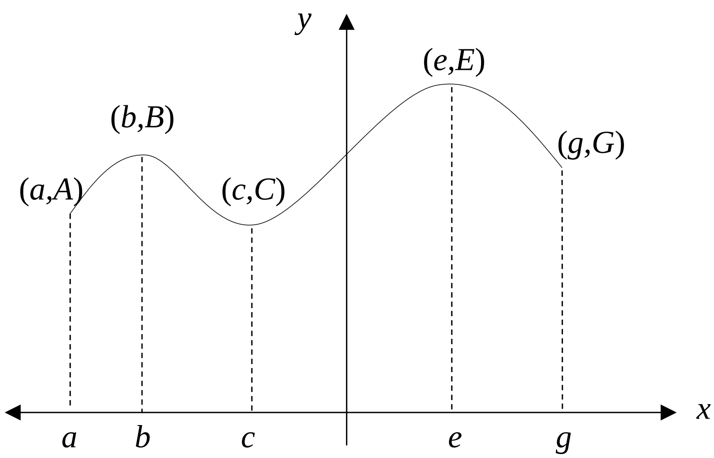

Figure 1. Sketch showing local and absolute minimums and maximums for a function.

Look at Figure 1 where if the domain D is given by the interval $[a,g]$, then C= absolute minimum of the function, as it is the smallest value of the function in the domain D. To find the absolute minimum of a continuous function with domain D, we look at the value of the function at the end points of D and also check where $f'(x)=0$.

These points where $f'(x)=0$ are the points of local extreme (local minimum or maximum) values. If $f''(x)>0$ at any of these points where local extremes occur, then it corresponds to a local minimum. Out of all the local minimums and the values at the domain ends, one can find the minimum value. The point where this minimum exists is then the location of the absolute minimum value, and the value of the function at that point is the absolute minimum.

Example

Find the location of the minimum of a polynomial $25-20x+4x^2$.

Solution

$$f'(x) = \frac{d}{dx}(25-20x+4x^2)$$

$$= \frac{d}{dx}(25)+\frac{d}{dx}(-20x)+\frac{d}{dx}(4x^2)$$

$$= 0-20+4\frac{d}{dx}(x^2)$$

$$= -20x+4(2x)$$

$$= -20+8x$$

Check where $f'(x)=0$

$$-20+8x=0$$

$$x=\frac{20}{8}$$

$$=2.5$$

Now check for $f''(x)$

$$f'(x)=-20+8x$$

$$f''(x)=\frac{d}{dx}(-20+8x)$$

$$=8$$

$$f''(2.5)=8$$

Since $f'(2.5)=0$ and $f''(2.5)>0$, the function has a local minimum at $x=2.5$. Is it also the location of the absolute minimum? Yes, because of two reasons – firstly, the value of the function approaches infinity (maximum value) at the end points of the domain $(-\infty,\infty)$. That is the check to look for extremes at the end points of the domain. Secondly, there is only one point at which $f'(x)=0$ and $f''(x)>0$.

Example

Given (x, y) data points (5,10), (6,15), (10,20), find minimum of the summation series, where a is a variable.

$$S = \sum_{i=1}^{3} (y_i - ax_i)^2$$

Solution

The *(x,y)* data pairs are given

$$x_1 = 5, x_2 = 6, x_3 = 10, y_1 = 10, y_2 = 15, y_3 = 20$$

$$S = \sum_{i=1}^{3} (y_i - ax_i)^2$$

$$= (y_1 - ax_1)^2 + (y_2 - ax_2)^2 + (y_3 - ax_3)^2$$

$$= (10 - 5a)^2 + (15 - 6a)^2 + (20 - 10a)^2$$

$$= 161a^2 - 680a + 725$$

$$\frac{dS}{da} = \frac{d}{da}(161a^2 - 680a + 725)$$

$$= 161(2a) - 680$$

$$= 322a - 680$$

$$\frac{dS}{da} = 0$$

$$322a - 680 = 0$$

$$a = \frac{680}{322}$$

$$= 2.111$$

$$\frac{d^2S}{da^2} = \frac{d}{da}\left(\frac{dS}{da}\right)$$

$$= \frac{d}{da}(322a - 680)$$

$$= 322$$

$$\frac{d^2S}{da^2}(2.11) = 322$$

Hence,

$$\frac{dS}{da} = 0 \text{ at } a\text{=2.11 and}$$

$$\frac{d^2S}{da^2}(2.11) > 0\,.$$

A local minimum exists at a=2.11. Since S is a continuous function and it has only one point where $\frac{dS}{da} = 0$ and $\frac{d^2S}{da^2} > 0$, it corresponds to a local minimum as well as the absolute minimum.

Alternative Solution

Look at the solution if we had not expanded the summation.

$$S = \sum_{i=1}^{n} (y_i - ax_i)^2$$

Using the chain rule, if u is a function of variable a

$$\frac{d}{da}(u^2) = 2u\frac{du}{da}$$

then

$$\frac{dS}{da} = 0$$

gives

$$\sum_{i=1}^{n} 2(y_i - ax_i)(-x_i) = 0$$

$$\sum_{i=1}^{n} -2y_i x_i + 2ax_i^{\ 2} = 0$$

$$\sum_{i=1}^{3} -2y_i x_i + \sum_{i=1}^{3} 2ax_i^{\ 2} = 0$$

$$-2\sum_{i=1}^{3} y_i x_i + 2a\sum_{i=1}^{3} x_i^{\ 2} = 0$$

$$2a\sum_{i=1}^{3} x_i^{\ 2} = 2\sum_{i=1}^{3} y_i x_i$$

$$a = \frac{\sum_{i=1}^{3} y_i x_i}{\sum_{i=1}^{3} x_i^{\ 2}}$$

$$= \frac{(10\times 5)+(15\times 6)+(20\times 10)}{5^2+6^2+10^2}$$

$$= \frac{340}{161}$$

$$= 2.11$$

We found

$$\frac{dS}{da} = \sum_{i=1}^{3}(-2y_i x_i + 2ax_i^{\ 2})$$

then

$$\frac{d^2S}{da^2} = \sum_{i=1}^{3} 2x_i^{\ 2}$$

$$= 2(5)^2 + 2(6)^2 + 2(10)^2$$

$$= 322$$

A local minimum exists at *a*=2.11. Since S is a continuous function and has only one point where $\frac{dS}{da} = 0$ and $\frac{d^2S}{da^2} > 0$, it corresponds to a local minimum as well as the absolute minimum.