**Chapter 07.05**
**Gaussian Quadrature Rule of Integration**

## Lesson: Higher point Gauss quadrature formulas

After successful completion of this lesson, you should be able to:
1) apply higher point Gauss quadrature formulas to estimate integrals
2) use table of abscissas and weights from a table to apply Gauss quadrature rule.

We have discussed one-point and two-point Gauss quadrature rules in a previous lesson. So, how do higher point Gauss quadrature rules work. For example, the three-point Gauss quadrature rule is given by

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) \quad (17)$$

The coefficients $c_1$, $c_2$ and $c_3$, and the function arguments $x_1$, $x_2$ and $x_3$ are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5)dx.$$

General $n$ -point rules would approximate an integral as

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots + c_n f(x_n) \quad (18)$$

**Arguments and weighing factors for *n*-point Gauss quadrature rules**
In handbooks (see Table 1), coefficients and arguments for $n$-point Gauss quadrature rule are not given for integrals of the form $\int_a^b f(x)dx$ but for integrals of the form $\int_{-1}^{1} g(x)dx$, that is,

$$\int_{-1}^{1} g(x)dx \approx \sum_{i=1}^{n} c_i g(x_i) \quad (19)$$

**Table 1** Weighting factors $c$ and function arguments $x$ used in Gauss quadrature formulas

| Points | Weighting Factors | Function Arguments |
|---|---|---|
| 2 | $c_1 = 1.000000000$ | $x_1 = -0.577350269$ |
| | $c_2 = 1.000000000$ | $x_2 = 0.577350269$ |
| 3 | $c_1 = 0.555555556$ | $x_1 = -0.774596669$ |
| | $c_2 = 0.888888889$ | $x_2 = 0.000000000$ |
| | $c_3 = 0.555555556$ | $x_3 = 0.774596669$ |
| 4 | $c_1 = 0.347854845$ | $x_1 = -0.861136312$ |
| | $c_2 = 0.652145155$ | $x_2 = -0.339981044$ |
| | $c_3 = 0.652145155$ | $x_3 = 0.339981044$ |
| | $c_4 = 0.347854845$ | $x_4 = 0.861136312$ |

| | | |
|---|---|---|
| 5 | $c_1 = 0.236926885$<br>$c_2 = 0.478628670$<br>$c_3 = 0.568888889$<br>$c_4 = 0.478628670$<br>$c_5 = 0.236926885$ | $x_1 = -0.906179846$<br>$x_2 = -0.538469310$<br>$x_3 = 0.000000000$<br>$x_4 = 0.538469310$<br>$x_5 = 0.906179846$ |
| 6 | $c_1 = 0.171324492$<br>$c_2 = 0.360761573$<br>$c_3 = 0.467913935$<br>$c_4 = 0.467913935$<br>$c_5 = 0.360761573$<br>$c_6 = 0.171324492$ | $x_1 = -0.932469514$<br>$x_2 = -0.661209386$<br>$x_3 = -0.238619186$<br>$x_4 = 0.238619186$<br>$x_5 = 0.661209386$<br>$x_6 = 0.932469514$ |

**So if the table is given for $\int_{-1}^{1} g(x)dx$ integrals, how does one solve $\int_{a}^{b} f(x)dx$?**

The answer lies in that any integral with limits of $[a, b]$ can be converted into an integral with limits $[-1,1]$. Let

$$x = mt + c \tag{20}$$

If $x = a$, then $t = -1$
If $x = b$, then $t = +1$
such that

$$a = m(-1) + c$$
$$b = m(1) + c \tag{21}$$

Solving the two Equations (21) simultaneously gives

$$m = \frac{b-a}{2}$$
$$c = \frac{b+a}{2} \tag{22}$$

Hence

$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$
$$dx = \frac{b-a}{2}dt$$

Substituting our values of $x$ and $dx$ into the integral gives us

$$\int_{a}^{b} f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2}dx \tag{23}$$

Hence any integral of the form $\int_{a}^{b} f(x)dx$ can be converted to an $\int_{-1}^{1} g(x)dx$, and hence Table 1 can be used to estimate integrals.

**Example 4**

Use two-point Gauss quadrature rule to approximate the distance covered by a rocket from $t = 8$ to $t = 30$ as given by

$$x = \int_8^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right) dt$$

Change the limits so that one can use the weights and abscissas given in Table 1. Also, find the absolute relative true error.

**Solution**

First, change the limits of integration from [8,30] to $[-1,1]$ using Equation(23) gives

$$\int_8^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right) dx$$

$$= 11\int_{-1}^{1} f(11x + 19)dx$$

Next, get weighting factors and function argument values from Table 1 for the two-point rule,

$c_1 = 1.000000000.$
$x_1 = -0.577350269$
$c_2 = 1.000000000$
$x_2 = 0.577350269$

Now we can use the Gauss quadrature formula

$$11\int_{-1}^{1} f(11x+19)dx \approx 11[c_1 f(11x_1 + 19) + c_2 f(11x_2 + 19)]$$

$$= 11[f(11(-0.5773503) + 19) + f(11(0.5773503) + 19)]$$
$$= 11[f(12.64915) + f(25.35085)]$$
$$= 11[(296.8317) + (708.4811)]$$
$$= 11058.44m$$

since

$$f(12.64915) = 2000\, ln\left[\frac{140000}{140000 - 2100(12.64915)}\right] - 9.8(12.64915)$$
$$= 296.8317$$

$$f(25.35085) = 2000\, ln\left[\frac{140000}{140000 - 2100(25.35085)}\right] - 9.8(25.35085)$$
$$= 708.4811$$

The absolute relative true error, $|\in_t|$, is (True value = 11061.34 m)

$$|\in_t| = \left|\frac{11061.34 - 11058.44}{11061.34}\right| \times 100$$
$$= 0.0262\%$$

**Example 5**

Use three-point Gauss quadrature rule to approximate the distance covered by a rocket from $t = 8$ to $t = 30$ as given by

$$x = \int_8^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right) dt$$

Change the limits so that one can use the weights and abscissas given in Table 1. Also, find the absolute relative true error.

**Solution**

First, change the limits of integration from [8,30] to $[-1,1]$ using Equation (23) gives

$$\int_{8}^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right)dx$$
$$= 11\int_{-1}^{1} f(11x+19)dx$$

The weighting factors and function argument values are

$c_1 = 0.555555556$
$x_1 = -0.774596669$
$c_2 = 0.888888889$
$x_2 = 0.000000000$
$c_3 = 0.555555556$
$x_3 = 0.774596669$

and the formula is

$$11\int_{-1}^{1} f(11x+19)dx \approx 11[c_1 f(11x_1+19) + c_2 f(11x_2+19) + c_3 f(11x_3+19)]$$

$$= 11\begin{bmatrix} 0.5555556 f(11(-.7745967)+19) + 0.8888889 f(11(0.0000000)+19) \\ +0.5555556 f(11(0.7745967)+19) \end{bmatrix}$$

$$= 11[0.55556 f(10.47944) + 0.88889 f(19.00000) + 0.55556 f(27.52056)]$$

$$= 11[0.55556 \times 239.3327 + 0.88889 \times 484.7455 + 0.55556 \times 795.1069]$$

$$= 11061.31m$$

since

$$f(10.47944) = 2000\, ln\left[\frac{140000}{140000-2100(10.47944)}\right] - 9.8(10.47944)$$
$$= 239.3327$$
$$f(19.00000) = 2000\, ln\left[\frac{140000}{140000-2100(19.00000)}\right] - 9.8(19.00000)$$
$$= 484.7455$$
$$f(27.52056) = 2000\, ln\left[\frac{140000}{140000-2100(27.52056)}\right] - 9.8(27.52056)$$
$$= 795.1069$$

The absolute relative true error, $|\in_t|$, is (True value = 11061.34 m)

$$|\in_t| = \left|\frac{11061.34 - 11061.31}{11061.34}\right| \times 100$$
$$= 0.0003\%$$

**So does Gaussian quadrature require that the integral must be transformed to the integral limit of [-1,1]?**

No, the limits do not need to be transformed. Gaussian quadrature rule can be written for any limits of integration.

It is just the weights and abscissas are given for the limits of integration of [-1,1]. So if the *n*-point Gaussian quadrature rule for [-1,1] limits is given as

$$\int_{-1}^{1} g(x)dx \approx \sum_{i=1}^{n} c_i g(x_i),$$

and we also know that

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

then the *n*-point Gaussian quadrature rule for [*a*,*b*] limits of integration can be also found.

$$\begin{aligned}\int_a^b f(x)dx &= \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx \\ &\approx \frac{b-a}{2}\sum_{i=1}^{n} c_i\, f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right) \\ &\approx \sum_{i=1}^{n} \frac{b-a}{2} c_i\, f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right) \\ &= \sum_{i=1}^{n} C_i f(X_i)\end{aligned}$$

where

$$C_i = \frac{b-a}{2} c_i$$

$$X_i = \frac{b-a}{2}x_i + \frac{b+a}{2}$$

**Appendix**

**Example 1**

For an integral $\int_{-1}^{1} f(x)dx$, derive the two-point Gauss quadrature rule

$$\int_{-1}^{1} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

where

$$c_1 = 1$$

$$c_2 = 1$$

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

**Solution**

Assuming the formula

$$\int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2)$$

(E1.1)

gives exact values for integrals $\int_{-1}^{1} 1dx$, $\int_{-1}^{1} xdx$, $\int_{-1}^{1} x^2dx$, and $\int_{-1}^{1} x^3dx$ . Then

$\int_{-1}^{1} 1dx = 2 = c_1 + c_2$

(E1.2)

$\int_{-1}^{1} xdx = 0 = c_1x_1 + c_2x_2$ (E1.3)

$\int_{-1}^{1} x^2dx = \frac{2}{3} = c_1{x_1}^2 + c_2{x_2}^2$

(E1.4)

$\int_{-1}^{1} x^3dx = 0 = c_1{x_1}^3 + c_2{x_2}^3$

(E1.5)

Multiplying Equation (E1.3) by ${x_1}^2$ and subtracting from Equation (E1.5) gives

$c_2x_2({x_1}^2 - {x_2}^2) = 0$ (E1.6)

The solution to the above equation is

$c_2 = 0$, or/and

$x_2 = 0$, or/and

$x_1 = x_2$, or/and

$x_1 = -x_2$.

I. $c_2 = 0$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 = 2$, $c_1x_1 = 0$, $c_1x_1^2 = \frac{2}{3}$, and $c_1x_1^3 = 0$. But since $c_1 = 2$, then $x_1 = 0$ from $c_1x_1 = 0$, but $x_1 = 0$ conflicts with $c_1x_1^2 = \frac{2}{3}$.

II. $x_2 = 0$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 + c_2 = 2$, $c_1x_1 = 0$, $c_1x_1^2 = \frac{2}{3}$, and $c_1x_1^3 = 0$. Since $c_1x_1 = 0$, then $c_1$ or $x_1$ has to be zero but this violates $c_1x_1^2 = \frac{2}{3} \neq 0$.

III. $x_1 = x_2$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 + c_2 = 2$, $c_1x_1 + c_2x_1 = 0$, $c_1x_1^2 + c_2{x_1}^2 = \frac{2}{3}$, and $c_1x_1^3 + c_2x_1^3 = 0$. If $x_1 \neq 0$, then $c_1x_1 + c_2x_1 = 0$ gives $c_1 + c_2 = 0$ and that violates $c_1 + c_2 = 2$. If $x_1 = 0$, then that violates $c_1x_1^2 + c_2{x_1}^2 = \frac{2}{3} \neq 0$.

That leaves the solution of $x_1 = -x_2$ as the only possible acceptable solution and in fact, it does not have violations (see it for yourself)

$x_1 = -x_2$ (E1.7)

Substituting (E1.7) in Equation (E1.3) gives

$c_1 = c_2$ (E1.8)

From Equations (E1.2) and (E1.8),

$c_1 = c_2 = 1$ (E1.9)

Equations (E1.4) and (E1.9) gives

${x_1}^2 + {x_2}^2 = \frac{2}{3}$ (E1.10)

Since Equation (E1.7) requires that the two results be of opposite sign, we get

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

Hence

$$\int_{-1}^{1} f(x)dx = c_1f(x_1) + c_2f(x_2) \quad \text{(E1.11)}$$

$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

**Example 3**

What would be the formula for

$$\int_a^b f(x)dx = c_1 f(a) + c_2 f(b)$$

if you want the above formula to give you exact values of integral $\int_a^b (a_0 x + b_0 x^2)dx$, that is, a linear combination of $x$ and $x^2$.

**Solution**

If the formula is exact for a linear combination of $x$ and $x^2$, then

$$\int_a^b x dx = \frac{b^2 - a^2}{2} = c_1 a + c_2 b$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 b^2 \qquad \text{(E3.1)}$$

Solving the two Equations (E3.1) simultaneously gives

$$\begin{bmatrix} a & b \\ a^2 & b^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{b^2 - a^2}{2} \\ \frac{b^3 - a^3}{3} \end{bmatrix}$$

$$c_1 = -\frac{1}{6} \frac{-ab - b^2 + 2a^2}{a}$$

$$c_2 = -\frac{1}{6} \frac{a^2 + ab - 2b^2}{b} \qquad \text{(E3.2)}$$

So

$$\int_a^b f(x)dx = -\frac{1}{6} \frac{-ab - b^2 + 2a^2}{a} f(a) - \frac{1}{6} \frac{a^2 + ab - 2b^2}{b} f(b) \qquad \text{(E3.3)}$$

Let us see if the formula works.

Evaluate $\int_2^5 (2x^2 - 3x)dx$ using Equation(E3.3)

$$\int_2^5 (2x^2 - 3x)dx \approx c_1 f(a) + c_2 f(b)$$

$$= -\frac{1}{6} \frac{-(2)(5) - 5^2 + 2(2)^2}{2} [2(2)^2 - 3(2)] - \frac{1}{6} \frac{2^2 + 2(5) - 2(5)^2}{5} [2(5)^2 - 3(5)]$$

$$= 46.5$$

The exact value of $\int_2^5 (2x^2 - 3x)dx$ is given by

$$\int_2^5 (2x^2 - 3x)\, dx = \left[\frac{2x^3}{3} - \frac{3x^2}{2}\right]_2^5$$

$$= 46.5$$

Any surprises?

Now evaluate $\int_2^5 3dx$ using Equation (E3.3)

$$\int_2^5 3dx \approx c_1 f(a) + c_2 f(b)$$

$$= -\frac{1}{6}\frac{-2(5)-5^2+2(2)^2}{2}(3) - \frac{1}{6}\frac{2^2+2(5)-2(5)^2}{5}(3)$$
$$= 10.35$$

The exact value of $\int_2^5 3dx$ is given by

$$\int_2^5 3dx = [3x]_2^5$$
$$= 9$$

Because the formula will only give exact values for linear combinations of $x$ and $x^2$, it does not work exactly even for a simple integral of $\int_2^5 3dx$.

Do you see now why we chose $a_0 + a_1 x$ as the integrand for which the formula

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

gives us exact values?