Bus Suspension System

Autar Kaw
Humberto Isaza

http://nm.MathForCollege.com
Transforming Numerical Methods Education for STEM Undergraduates

Bus Suspension System

http://nm.MathForCollege.com

Bus Suspension System

An Example to Show How to Reduce Coupled Differential Equations to a Set of First Order Differential Equations

Figure 1: A school bus.

Bus Suspension System (cont.)

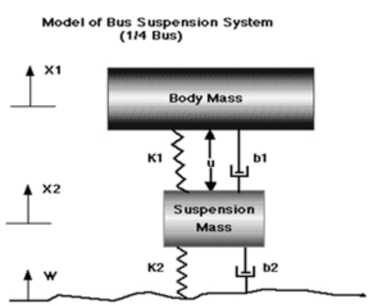


Figure 2: A model of 1/4th of suspension system of bus.

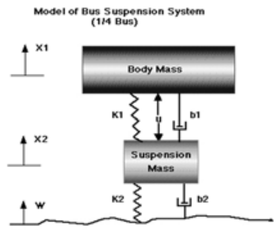
Bus Suspension System (cont.)

Problem Statement:
A suspension system of a bus can be modeled as above. Only 1/4th of the bus is modeled. The differential equations that govern the above system can be derived (this is something you will do in your vibrations course) as

$$M_1 \frac{d^2x_1}{dt^2} + B_1\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) + K_1(x_1 - x_2) = 0 \quad (1)$$

$$M_2 \frac{d^2x_2}{dt^2} + B_1\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + K_1(x_2 - x_1) + B_2\left(\frac{dx_2}{dt} - \frac{dw}{dt}\right) + K_2(x_2 - w) = 0 \quad (2)$$

$$x_1(0) = 0, v_1(0) = 0, x_2(0) = 0, v_2(0) = 0$$



Bus Suspension System (cont.)

Where

$M_1 =$ body
$M_2 =$ suspension mass
$K_1 =$ spring constant of suspension system
$K_2 =$ spring constant of wheel and tire
$B_1 =$ damping constant of suspension system
$B_2 =$ damping constant of wheel and tire
$x_1 =$ displacement of the body mass as a function of time
$x_2 =$ displacement of the suspension mass as a function of time
$w =$ input profile of the road as a function of time

Bus Suspension System (cont.)

The constants are given as

$m_1 = 2500\,\text{kg}$
$m_2 = 320\,\text{kg}$
$K_1 = 80{,}000\ \text{N/m}$
$K_2 = 500{,}000\ \text{N/m}$
$B_1 = 350\,\text{N-s/m}$
$B_2 = 15{,}020\ \text{N-s/m}$

Reduce the simultaneous differential equations (1) and (2) to simultaneous first order differential equations and put them in the state variable form complete with corresponding initial conditions.

Bus Suspension System (cont.)

Solution

Substituting the values of the constants in the two differential equations (1) and (2) gives the differential equations (3) and (4), respectively.

$$2500\frac{d^2x_1}{dt^2} + 350\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) + 80000(x_1 - x_2) = 0 \qquad (3)$$

$$320\frac{d^2x_2}{dt^2} + 350\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + 80000\,(x_2 - x_1) + 15020\left(\frac{dx_2}{dt} - \frac{dw}{dt}\right) + 500000\,(x_2 - w) = 0 \qquad (4)$$

Bus Suspension System (cont.)

Since w is an input, we take it to the right hand side to show it as a forcing function and rewrite Equation (4) as

$$320\frac{d^2x_2}{dt^2}+350\left(\frac{dx_2}{dt}-\frac{dx_1}{dt}\right)+80000\left(x_2-x_1\right)+15020\left(\frac{dx_2}{dt}\right)+500000\ x_2=15020\ \frac{dw}{dt}+500000\ w \quad (5)$$

Now let us start the process of reducing the 2 simultaneous differential equations {Equations (3) and (5)} to 4 simultaneous first order differential equations. Choose

$$\frac{dx_1}{dt}=v_1 \quad (6)$$

$$\frac{dx_2}{dt}=v_2 \quad (7)$$

Bus Suspension System (cont.)

then Equation (3)

$$2500\frac{d^2x_1}{dt^2}+350\left(\frac{dx_1}{dt}-\frac{dx_2}{dt}\right)+80000\left(x_1-x_2\right)=0$$

can be written as

$$2500\frac{dv_1}{dt}+350\left(v_1-v_2\right)+80000\left(x_1-x_2\right)=0$$

$$2500\frac{dv_1}{dt}=-350\left(v_1-v_2\right)-80000\left(x_1-x_2\right)$$

$$\frac{dv_1}{dt}=-0.14\left(v_1-v_2\right)-32\left(x_1-x_2\right) \quad (8)$$

Bus Suspension System (cont.)

and Equation (5)

$$320\frac{d^2x_2}{dt^2}+350\left(\frac{dx_2}{dt}-\frac{dx_1}{dt}\right)+80000\left(x_2-x_1\right)+15020\left(\frac{dx_2}{dt}\right)+500000\,x_2=15020\,\frac{dw}{dt}+500000\,w$$

can be written as

$$320\frac{dv_2}{dt}+350(v_2-v_1)+80000(x_2-x_1)+15020v_2+500000x_2=15020\frac{dw}{dt}+500000w$$

$$320\frac{dv_2}{dt}=-350(v_2-v_1)-80000(x_2-x_1)-15020v_2-500000x_2+15020\frac{dw}{dt}+500000w$$

$$\frac{dv_2}{dt}=-1.09375(v_2-v_1)-250(x_2-x_1)-46.9375v_2-1562.5x_2+46.9375\frac{dw}{dt}+1562.5w \qquad (9)$$

Bus Suspension System (cont.)

The 4 simultaneous first order differential equations given by Equations 6 thru 9 complete with the corresponding initial condition then are

$$\frac{dx_1}{dt}=v_1=f_1(t,x_1,x_2,v_1,v_2),\ \ x_1(0)=0 \qquad (10)$$

$$\frac{dx_2}{dt}=v_2=f_2(t,x_1,x_2,v_1,v_2),\ \ x_2(0)=0 \qquad (11)$$

$$\frac{dv_1}{dt}=-0.14(v_1-v_2)-32(x_1-x_2)=f_3(t,x_1,x_2,v_1,v_2),\ \ v_1(0)=0 \qquad (12)$$

$$\frac{dv_2}{dt}=-1.09375(v_2-v_1)-250(x_2-x_1)-46.9375v_2-1562.5x_2+46.9375\frac{dw}{dt}+1562.5w$$
$$=f_4(t,x_1,x_2,v_1,v_2),\ \ v_2(0)=0 \qquad (13)$$

Bus Suspension System (cont.)

Assuming that the bus is going at 60 mph, that is, approximately 27 m/s, it takes

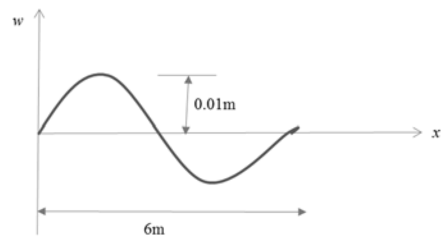
$$\frac{6m}{27m/s} = 0.22s$$

to go through one period. So the frequency

$$f = \frac{1}{0.22} = 4.545\,\text{Hz}$$

The angular frequency then is

$$\omega = 2 \times \pi \times 4.545 = 28.6 rad/s$$



Bus Suspension System (cont.)

Giving

$$w = 0.01\sin(\omega t)$$

$$= 0.01\sin(28.6t)$$

and

$$\frac{dw}{dt} = 0.286\cos(28.6t)$$

Bus Suspension System (cont.)

To put the differential equations given by Equations (10)-(13) in matrix form, we rewrite them as

$$\frac{dx_1}{dt} = v_1 = 1v_1 + 0v_2 + 0x_1 + 0x_2, \quad x_1(0) = 0 \qquad (10)$$

$$\frac{dx_2}{dt} = v_2 = 0v_1 + 1v_2 + 0x_1 + 0x_2, \quad x_2(0) = 0 \qquad (11)$$

$$\frac{dv_1}{dt} = -0.14v_1 + 0.14v_2 - 32x_1 + 32x_2, \quad v_1(0) = 0 \qquad (12)$$

$$\frac{dv_2}{dt} = 250x_1 - 1812.5x_2 + 1.09375v_1 - 48.03125v_2 + 1562.5w + 46.9375\frac{dw}{dt} \qquad (13)$$
$$= f_4(t, x_1, x_2, v_1, v_2), \quad v_2(0) = 0$$

Bus Suspension System (cont.)

In state variable matrix form, the differential equations are given by

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -32 & 32 & -0.14 & 0.14 \\ 250 & -1812.5 & 1.09375 & -48.03125 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1562.5w + 46.9375\frac{dw}{dt} \end{bmatrix}$$

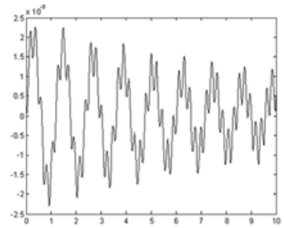
where $w = 0.01\sin(28.6t)$

and the corresponding initial conditions are

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ v_1(0) \\ v_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using MATLAB to Solve

```
function diffx=sysofeqn(t,x)
    A=[0  0  1  0; ...
       0  0  0  1; ...
      -32  32  -0.14  0.14; ...
      250   -1812.5  1.09375  -48.03125];
    w=0.01*sin(28.6*t);
    dw=0.286*cos(28.6*t);
    B=[0;  0;  0;   1652.5*w+46.9375*dw];
    diffx=A*x+B;
end
```



```
tspan=[0  10];
initial_cond=[0;0;0;0];
[t,x]=ode45('sysofeqn',tspan,initial_cond);
figure (1)
plot(t,x(:,1))
```

How the output data looks like

```
t =

                 0
   0.000003742346606
   0.000007484693212
   0.000011227039818
   0.000014969386423
   0.000033681119453
   0.000052392852482


x =

                 0                 0                 0                 0
   0.000000000000000   0.000000000094002   0.000000000013164   0.000050236522812
   0.000000000000000   0.000000000376002   0.000000000052670   0.000100470633045
   0.000000000000000   0.000000000845991   0.000000000118540   0.000150702329280
   0.000000000000001   0.000000001503960   0.000000000210794   0.000200931610101
   0.000000000000012   0.000000007613185   0.000000001068580   0.000452041733437
   0.000000000000045   0.000000018420550   0.000000002589166   0.000703091259071
```

End

See how MATLAB *ode45* is used to solve such problems

http://www.math.purdue.edu/academic/files/courses/past//2004fall/MA266/ode45.pdf