Bus Suspension System

An Example to Show How to Reduce Coupled Differential Equations to a Set of First Order Differential Equations

Figure 1: A school bus.

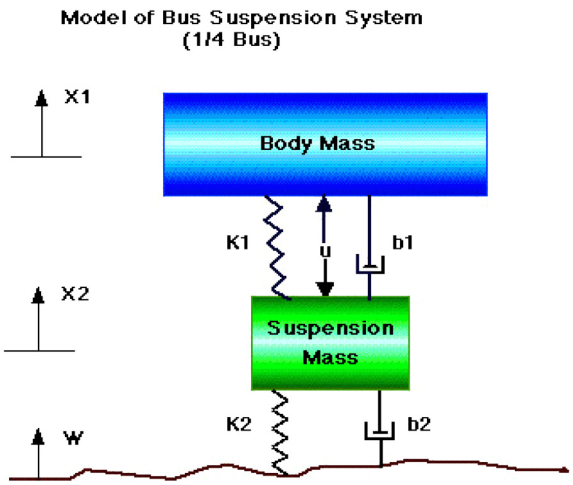

Figure 2: A model of 1/4th of suspension system of bus.

Problem Statement: A suspension system of a bus can be modeled as above. Only 1/4th of the bus is modeled.[1] The differential equations that govern the above system can be derived (this is something you will do in your vibrations course) as

$$M_1 \frac{d^2 x_1}{dt^2} + B_1\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) + K_1(x_1 - x_2) = 0 \qquad (1)$$

$$M_2 \frac{d^2 x_2}{dt^2} + B_1\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + K_1(x_2 - x_1) + B_2\left(\frac{dx_2}{dt} - \frac{dw}{dt}\right) + K_2(x_2 - w) = 0 \qquad (2)$$

$$x_1(0) = 0, v_1(0) = 0, x_2(0) = 0, v_2(0) = 0$$

where

$M_1 =$ body mass

$M_2 =$ suspension mass

$K_1 =$ spring constant of suspension system

$K_2 =$ spring constant of wheel and tire

$B_1 =$ damping constant of suspension system

$B_2 =$ damping constant of wheel and tire

$x_1 =$ displacement of the body mass as a function of time

$x_2 =$ displacement of the suspension mass as a function of time

$w =$ input profile of the road as a function of time

The constants are given as

$m_1 = 2500$ kg

$m_2 = 320$ kg

$k_1 = 80000$ N/m,

$k_2 = 500000$ N/m,

$B_1 = 350$ N-s/m,

$B_2 = 15020$ N-s/m

Reduce the simultaneous differential equations (1) and (2) to simultaneous first order differential equations and put those in the state variable form complete with corresponding initial conditions.

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[1] Model and numbers have been used from http://www.monografias.com/trabajos-pdf/modeling-bus-suspension-transfer-function/modeling-bus-suspension-transfer-function.pdf

Solution

Substituting the values of the constants in the two differential equations (1) and (2) gives the differential equations (3) and (4), respectively.

$$2500\frac{d^2x_1}{dt^2}+350\left(\frac{dx_1}{dt}-\frac{dx_2}{dt}\right)+80000(x_1-x_2)=0 \qquad (3)$$

$$320\frac{d^2x_2}{dt^2}+350\left(\frac{dx_2}{dt}-\frac{dx_1}{dt}\right)+80000(x_2-x_1)+15020\left(\frac{dx_2}{dt}-\frac{dw}{dt}\right)+500000(x_2-w)=0$$

(4)

Since w is an input, we take it to the right hand side to show it as a forcing function and rewrite Equations (4) as

$$320\frac{d^2x_2}{dt^2}+350\left(\frac{dx_2}{dt}-\frac{dx_1}{dt}\right)+80000(x_2-x_1)+15020\left(\frac{dx_2}{dt}\right)+500000x_2 =15020\frac{dw}{dt}+500000w \qquad (5)$$

Now let us start the process of reducing the 2 simultaneous differential equations (Equations (3) and (5)) to 4 simultaneous first order differential equations.

Choose

$$\frac{dx_1}{dt}=v_1, \qquad (6)$$

$$\frac{dx_2}{dt}=v_2, \qquad (7)$$

then Equation (3)

$$2500\frac{d^2x_1}{dt^2}+350\left(\frac{dx_1}{dt}-\frac{dx_2}{dt}\right)+80000(x_1-x_2)=0$$

can be written as

$$2500\frac{dv_1}{dt}+350(v_1-v_2)+80000(x_1-x_2)=0$$

$$2500\frac{dv_1}{dt}=-350(v_1-v_2)-80000(x_1-x_2)$$

$$\frac{dv_1}{dt}=-0.14(v_1-v_2)-32(x_1-x_2)$$

$$=-32x_1-0.14v_1+32x_2+0.14v_2 \qquad (8)$$

and Equation (5)

$$320\frac{d^2x_2}{dt^2}+350\left(\frac{dx_2}{dt}-\frac{dx_1}{dt}\right)+80000(x_2-x_1)+15020\left(\frac{dx_2}{dt}\right)+500000x_2$$

$$=15020\frac{dw}{dt}+500000w$$

can be written as

$$320\frac{dv_2}{dt}+350(v_2-v_1)+80000(x_2-x_1)+15020v_2+500000x_2=15020\frac{dw}{dt}+500000w$$

$$320\frac{dv_2}{dt}=-350(v_2-v_1)-80000(x_2-x_1)-15020v_2-500000x_2+15020\frac{dw}{dt}+500000w$$

$$\frac{dv_2}{dt}=-1.09375(v_2-v_1)-250(x_2-x_1)-46.9375v_2-1562.5x_2+46.9375\frac{dw}{dt}+1562.5w$$

$$=250x_1+1.09375v_1-1812.5x_2-48.03125v_2+1562.5w+46.9375\frac{dw}{dt}$$

(9)

The 4 simultaneous first order differential equations given by Equations 6 thru 9 complete with the corresponding initial condition then are

$$\frac{dx_1}{dt} = v_1$$
$$= f_1(t, x_1, x_2, v_1, v_2) \text{ with } x_1(0) = 0 \quad (10)$$

$$\frac{dx_2}{dt} = v_2$$
$$= f_2(t, x_1, x_2, v_1, v_2) \text{ with } x_2(0) = 0 \quad (11)$$

$$\frac{dv_1}{dt} = -32x_1 + 32x_2 - 0.14v_1 + 0.14v_2$$
$$= f_3(t, x_1, x_2, v_1, v_2) \text{ with } v_1(0) = 0$$
(12)

$$\frac{dv_2}{dt} = 250x_1 - 1812.5x_2 + 1.09375v_1 - 48.03125v_2 + 1562.5w + 46.9375\frac{dw}{dt} \quad (13)$$
$$= f_4(t, x_1, x_2, v_1, v_2) \text{ with } v_2(0) = 0$$

The profile of the road is given below.

Assuming that the bus is going at 60 mph, that is, approximately 27 m/s, it takes

$$\frac{6\text{m}}{27\text{m/s}} = 0.22s$$

to go through one period. So the frequency

$$\begin{aligned} f &= \frac{1}{0.22} \\ &= 4.545\text{ Hz} \end{aligned}$$

The angular frequency then is

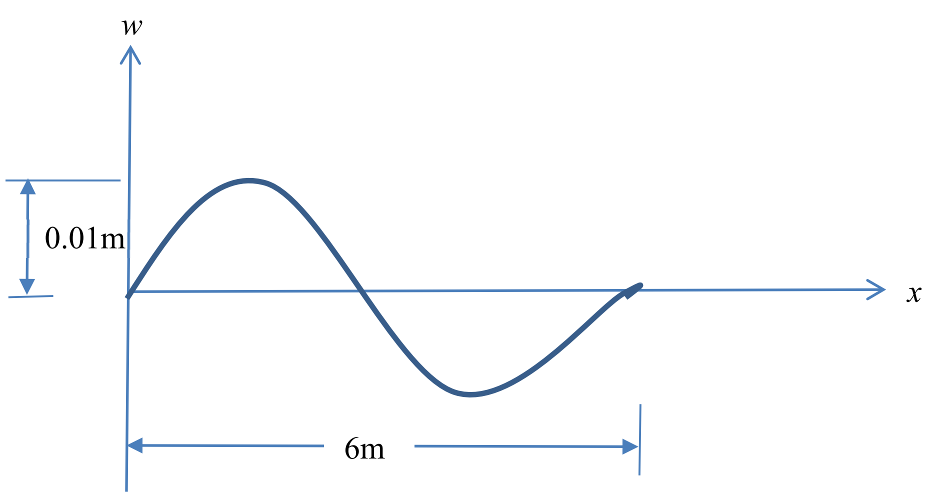
$$\begin{aligned} \omega &= 2 \times \pi \times 4.545 \\ &= 28.6 \, rad/s \end{aligned}$$

giving

$$\begin{aligned} w &= 0.01 \sin(\omega t) \\ &= 0.01 \sin(28.6t) \end{aligned}$$

and

$$\frac{dw}{dt} = 0.286 \cos(28.6t)$$



To put the differential equations given by Equations (10)-(13) in matrix form, we rewrite them as

$$\frac{dx_1}{dt} = v_1 = 1v_1 + 0v_2 + 0x_1 + 0x_2,\ x_1(0) = 0 \tag{14}$$

$$\frac{dx_2}{dt} = v_2 = 0v_1 + 1v_2 + 0x_1 + 0x_2,\ x_2(0) = 0 \tag{15}$$

$$\frac{dv_1}{dt} = -32x_1 + 32x_2 - 0.14v_1 + 0.14v_2,\ \ v_1(0) = 0 \tag{16}$$

$$\begin{aligned}\frac{dv_2}{dt} &= 250x_1 - 1812.5x_2 + 1.09375v_1 - 48.03125v_2 + 1562.5w + 46.9375\frac{dw}{dt} \\ &= f_4(t, x_1, x_2, v_1, v_2),\ \ v_2(0) = 0\end{aligned} \tag{17}$$

In state variable matrix form, the differential equations are given by

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -32 & 32 & -0.14 & 0.14 \\ 250 & -1812.5 & 1.09375 & -48.03125 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1562.5w + 46.9375\frac{dw}{dt} \end{bmatrix}$$

where

$$w = 0.01\sin(28.6t)$$

and the corresponding initial conditions are

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ v_1(0) \\ v_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

See how MATLAB *ode45* is used to solve such problems

http://www.math.purdue.edu/academic/files/courses/past//2004fall/MA266/ode45.pdf