

Interpolation

Reading Between the Lines

WHAT IS INTERPOLATION ?

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, find the value of 'y' at a value of 'x' that is not given.

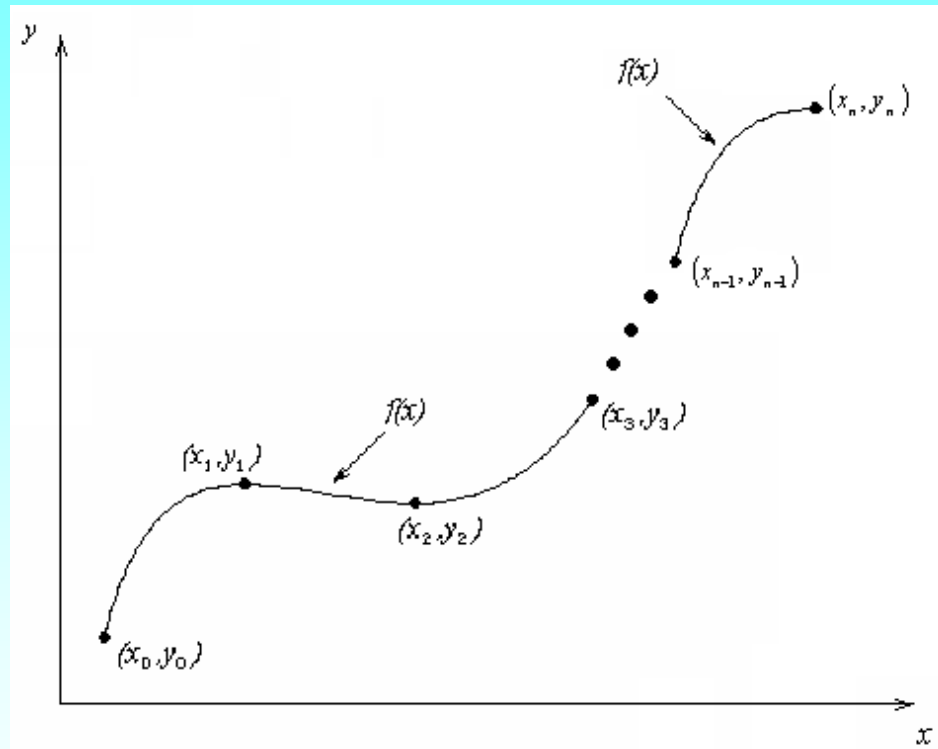


Figure Interpolation of discrete data.

APPLIED PROBLEMS

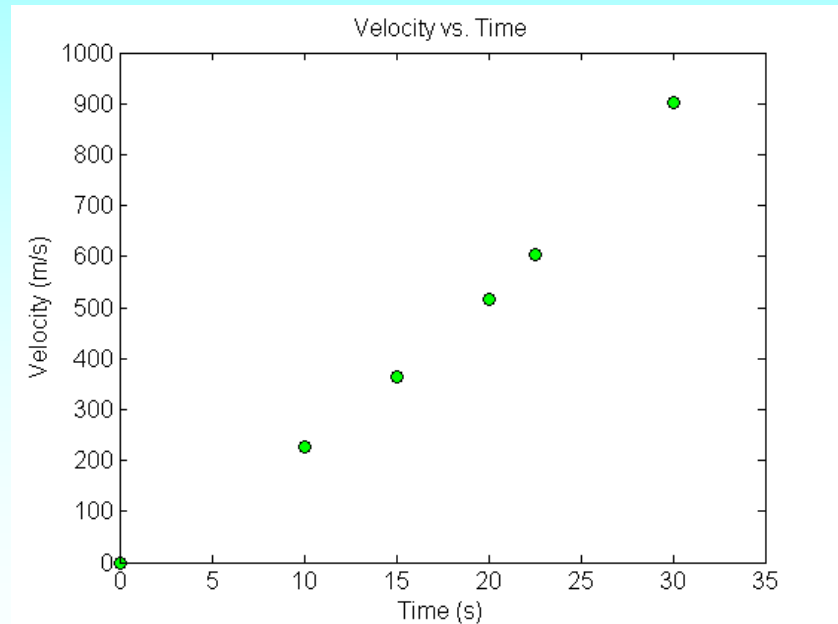
FLY ROCKET FLY, FLY ROCKET FLY



The upward velocity of a rocket is given as a function of time in table below. Find the velocity and acceleration at $t=16$ seconds.

Table Velocity as a function of time.

$t, (s)$	$v(t), (m/s)$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

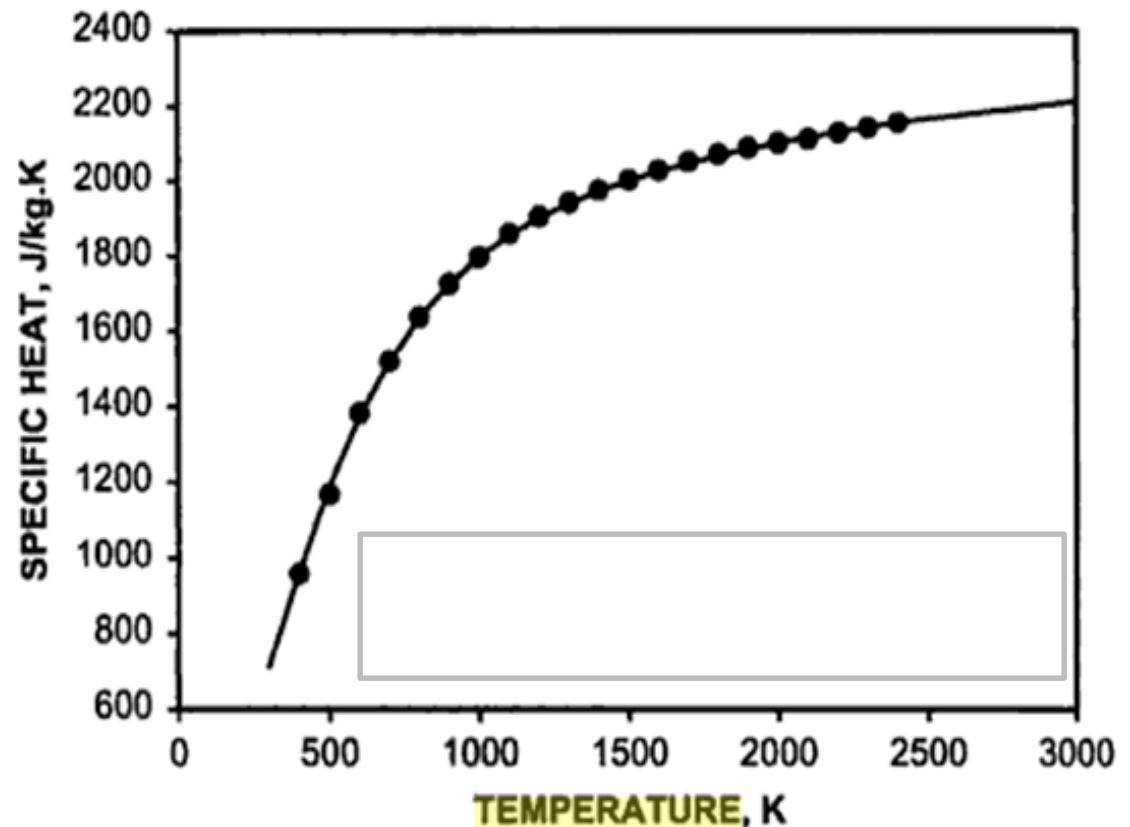


Velocity vs. time data for the rocket example

SPECIFIC HEAT OF CARBON

A carbon block is heated up from room temperature of 300K to 1800K.
How much heat is required to do so?

Temperature (K)	Specific Heat (J/kg-K)
200	420
400	1070
600	1370
1000	1820
1500	2000
2000	2120

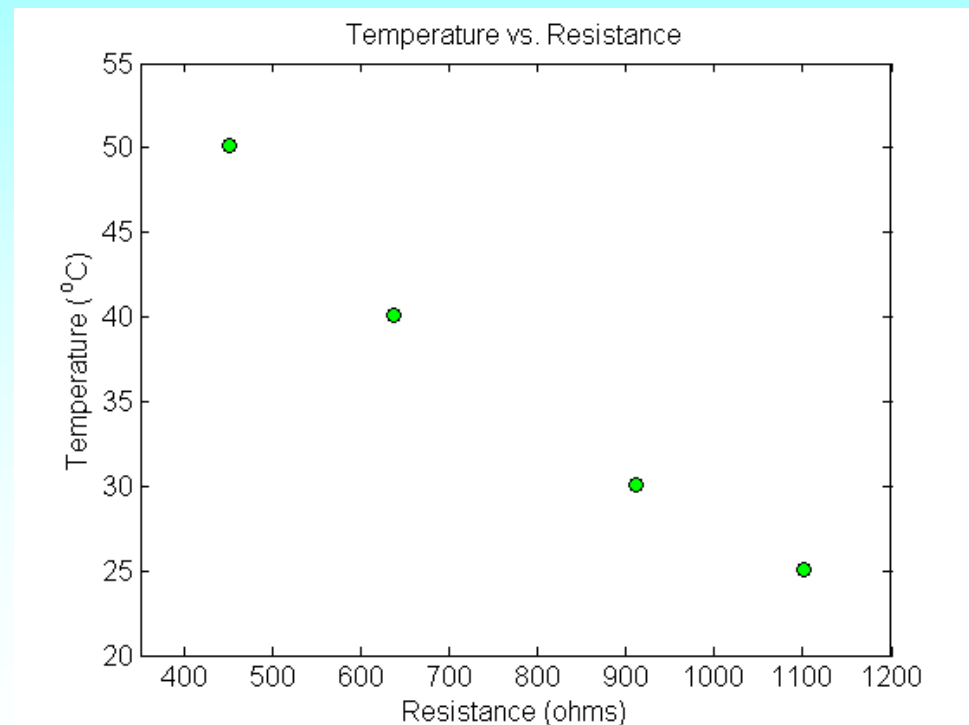
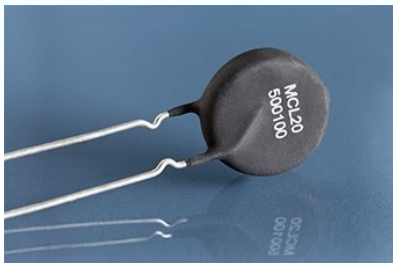


THERMISTOR CALIBRATION

Thermistors are based on change in resistance of a material with temperature. A manufacturer of thermistors makes the following observations on a thermistor. Determine the calibration curve for thermistor.

$$\frac{1}{T} = a_0 + a_1 \ln R + a_2 (\ln R)^2 + a_3 (\ln R)^3$$

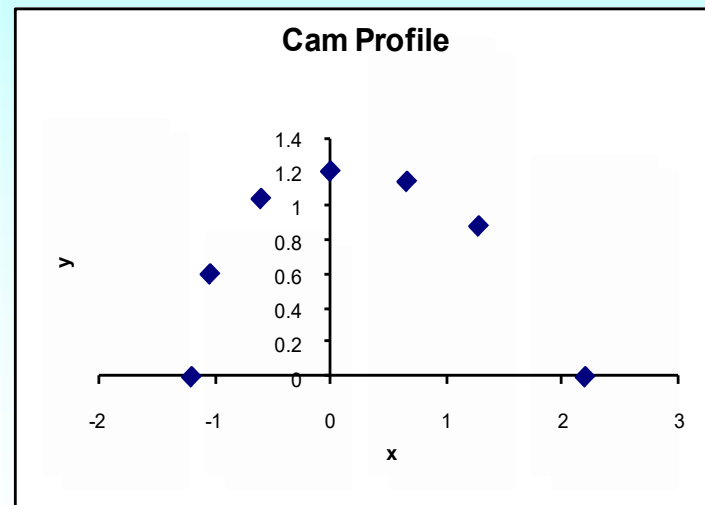
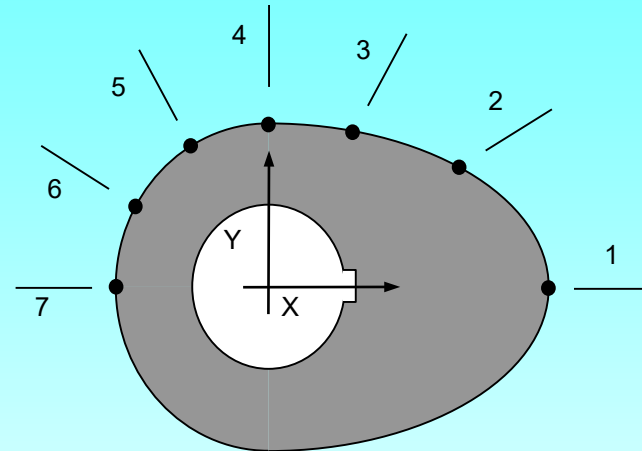
R (Ω)	T($^{\circ}\text{C}$)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128



FOLLOW THE CAM

A curve needs to be fit through the given points to fabricate the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	-0.60	1.04
6	-1.04	0.60
7	-1.20	0.00



Spline Interpolation Method

Major: All Engineering Majors

Authors: Autar Kaw, Jai Paul

<http://numericalmethods.eng.usf.edu>

Transforming Numerical Methods Education for STEM
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Spline Method of Interpolation

<http://nm.MathForCollege.com>

Why Splines ?

$$f(x) = \frac{1}{1 + 25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1 + 25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

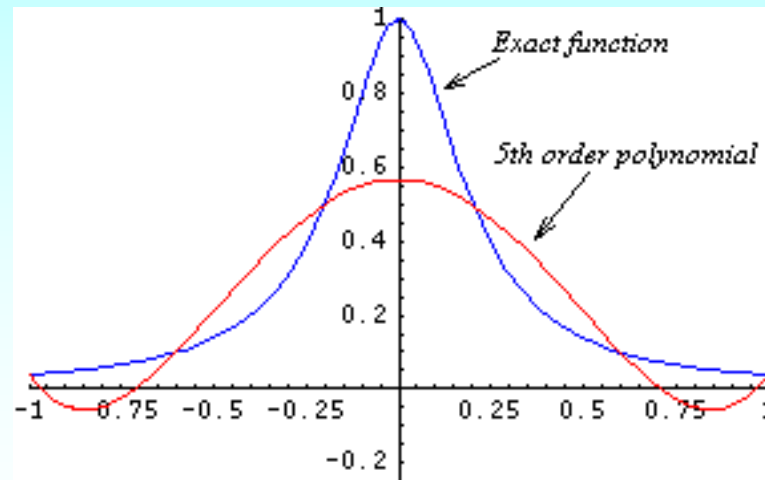


Figure : 5th order polynomial vs. exact function

Why Splines ?

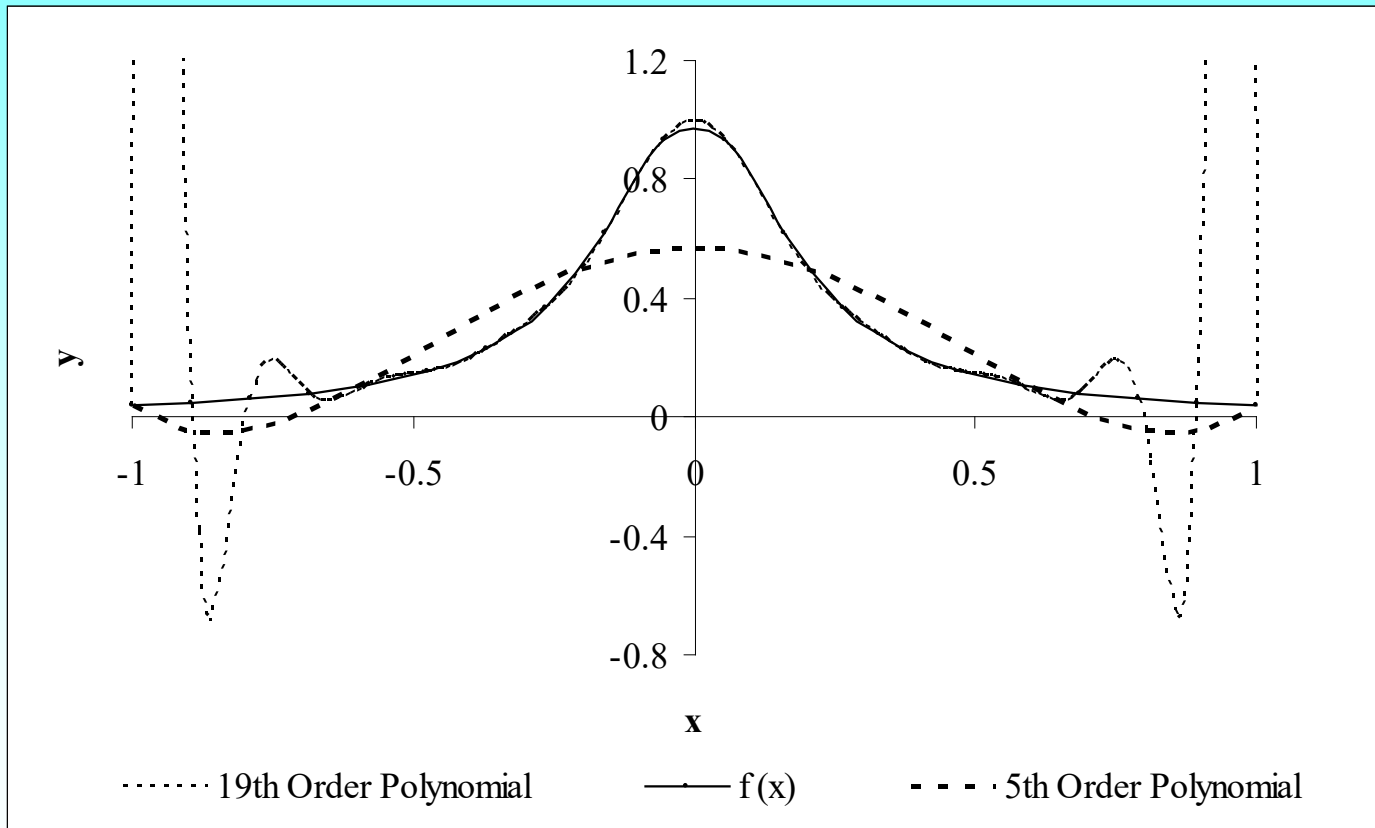
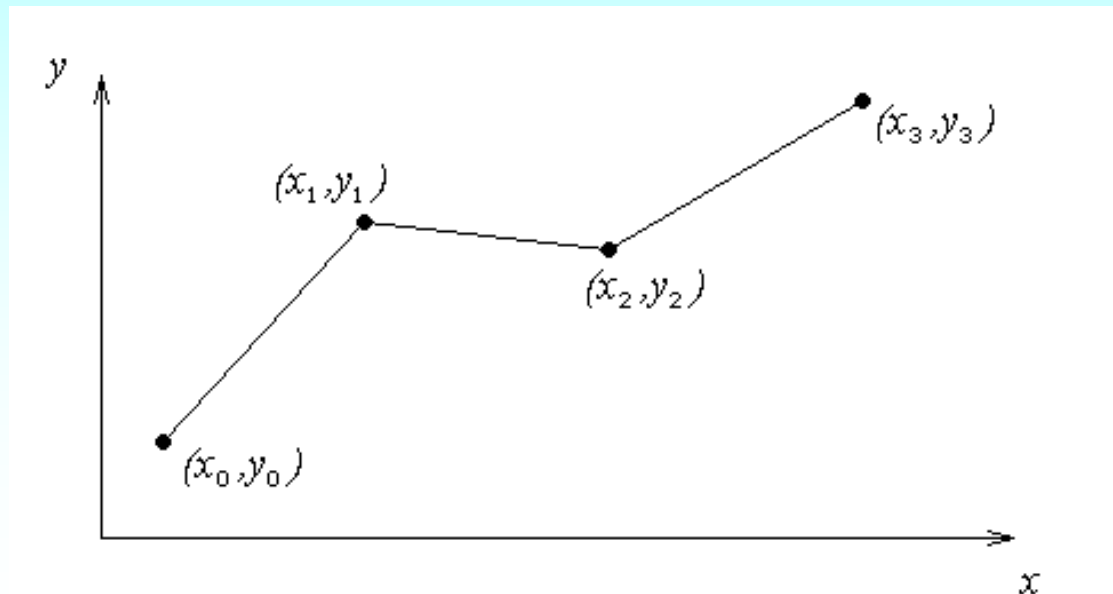


Figure : Higher order polynomial interpolation is a bad idea

Linear Spline Interpolation

Given $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $(y_i = f(x_i))$

Figure : Linear splines



Linear Spline Interpolation (contd)

$$\begin{aligned}f(x) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), & x_0 \leq x \leq x_1 \\&= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), & x_1 \leq x \leq x_2 \\&\vdots \\&\vdots \\&\vdots \\&= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), & x_{n-1} \leq x \leq x_n\end{aligned}$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between x_{i-1} and x_i .

Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at $t=16$ seconds using linear splines.

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

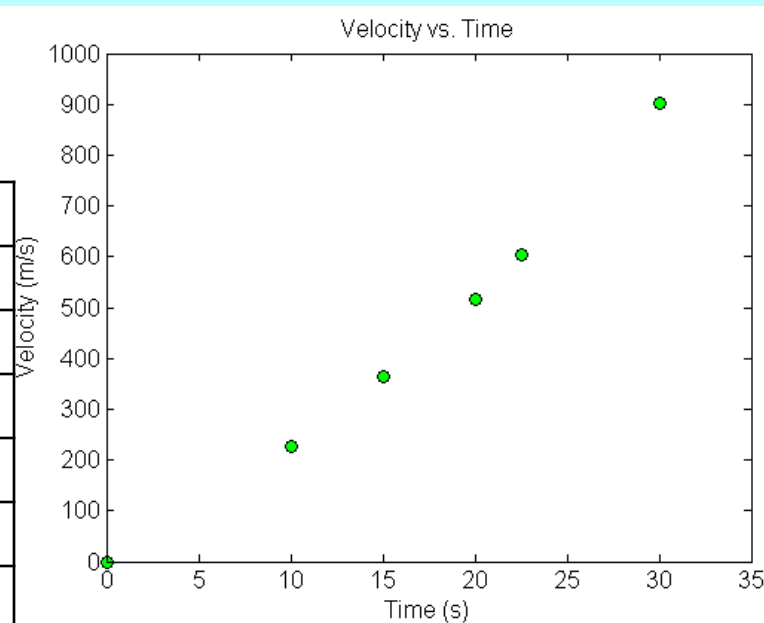


Figure. Velocity vs. time data for the rocket example



Linear Spline Interpolation

$$t_0 = 15, \quad v(t_0) = 362.78$$

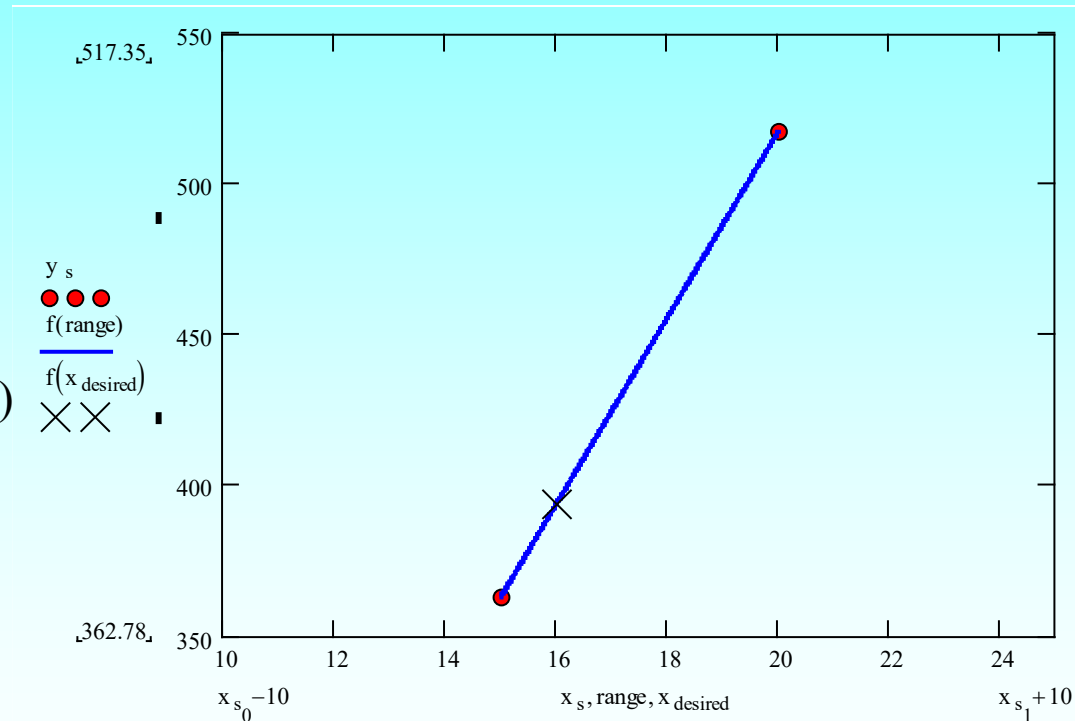
$$t_1 = 20, \quad v(t_1) = 517.35$$

$$\begin{aligned} v(t) &= v(t_0) + \frac{v(t_1) - v(t_0)}{t_1 - t_0} (t - t_0) \\ &= 362.78 + \frac{517.35 - 362.78}{20 - 15} (t - 15) \end{aligned}$$

$$v(t) = 362.78 + 30.913(t - 15)$$

At $t = 16$,

$$\begin{aligned} v(16) &= 362.78 + 30.913(16 - 15) \\ &= 393.7 \text{ m/s} \end{aligned}$$



Quadratic Spline Interpolation

Given $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$f(x) = a_1 x^2 + b_1 x + c_1, \quad x_0 \leq x \leq x_1$$

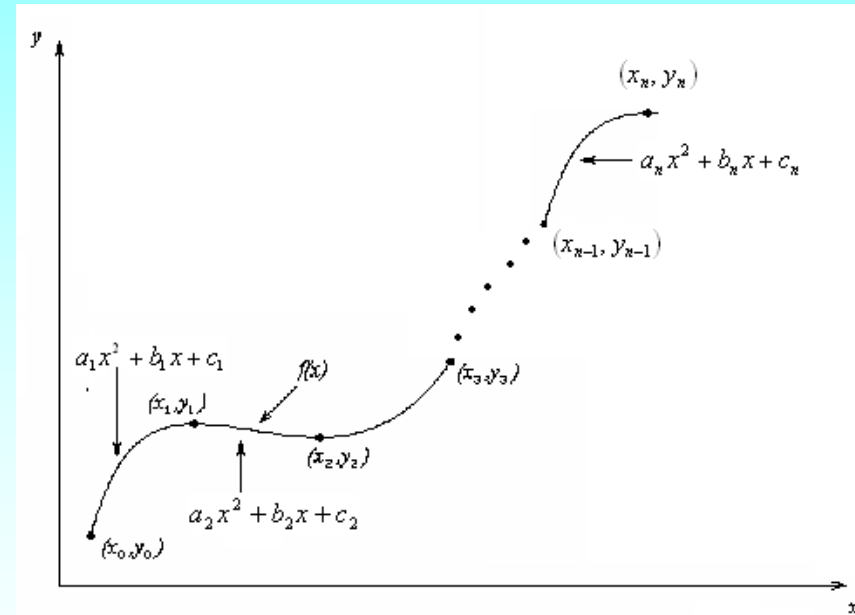
$$= a_2 x^2 + b_2 x + c_2, \quad x_1 \leq x \leq x_2$$

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$$= a_n x^2 + b_n x + c_n, \quad x_{n-1} \leq x \leq x_n$$

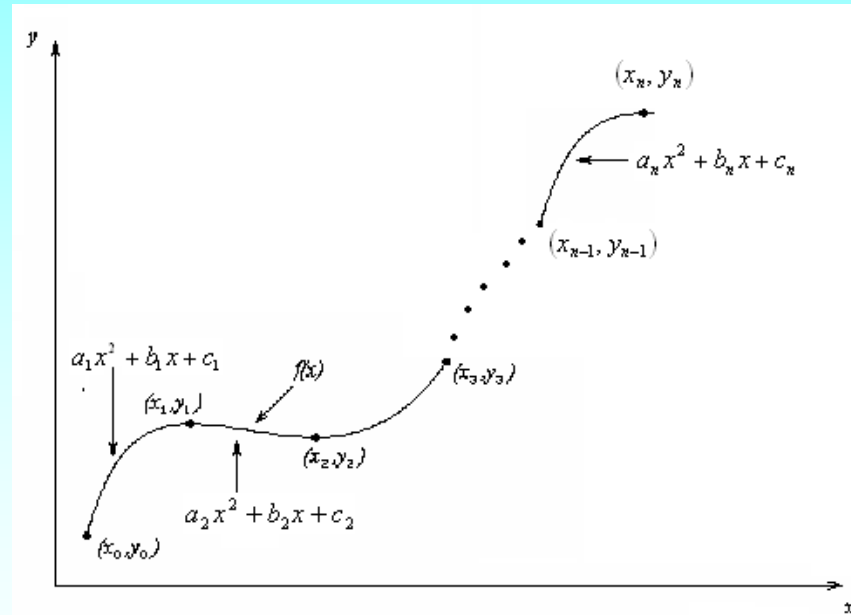


Find $a_i, b_i, c_i, i = 1, 2, \dots, n$

Quadratic Spline Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$\begin{aligned}
 a_1 x_0^2 + b_1 x_0 + c_1 &= f(x_0) \\
 a_1 x_1^2 + b_1 x_1 + c_1 &= f(x_1) \\
 &\vdots \\
 &\vdots \\
 a_i x_{i-1}^2 + b_i x_{i-1} + c_i &= f(x_{i-1}) \\
 a_i x_i^2 + b_i x_i + c_i &= f(x_i) \\
 &\vdots \\
 &\vdots \\
 a_n x_{n-1}^2 + b_n x_{n-1} + c_n &= f(x_{n-1}) \\
 a_n x_n^2 + b_n x_n + c_n &= f(x_n)
 \end{aligned}$$



This condition gives $2n$ equations

Quadratic Spline Interpolation (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \text{ is } 2a_1x + b_1$$

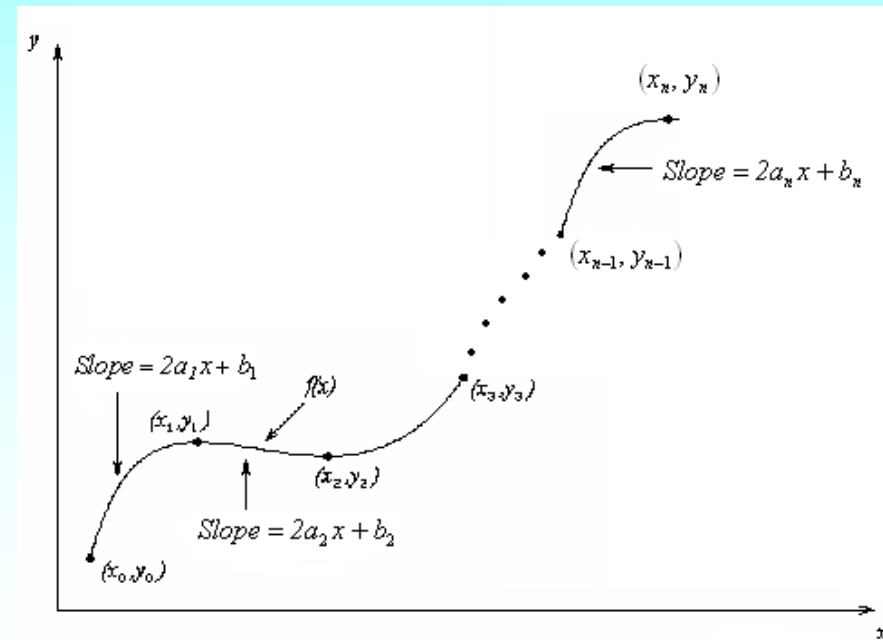
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \text{ is } 2a_2x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



Quadratic Spline Interpolation (contd)

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

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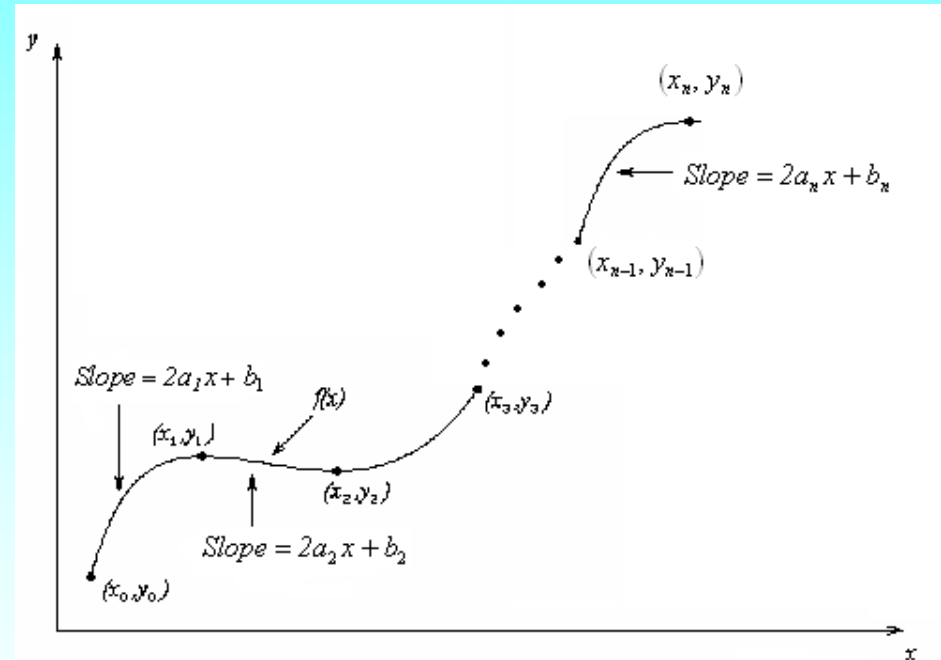
$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

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$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



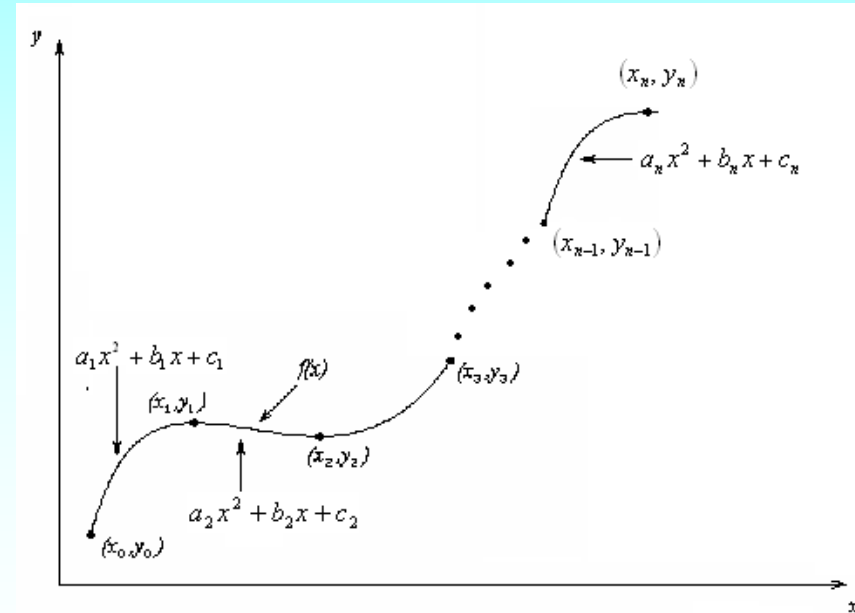
We have $(n-1)$ such equations. The total number of equations is $(2n) + (n-1) = (3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

Quadratic Spline Interpolation (contd)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

$$\begin{aligned}
 f(x) &= a_1x^2 + b_1x + c_1, & x_0 \leq x \leq x_1 \\
 &= a_2x^2 + b_2x + c_2, & x_1 \leq x \leq x_2 \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 &= a_nx^2 + b_nx + c_n, & x_{n-1} \leq x \leq x_n
 \end{aligned}$$



Quadratic Spline Example

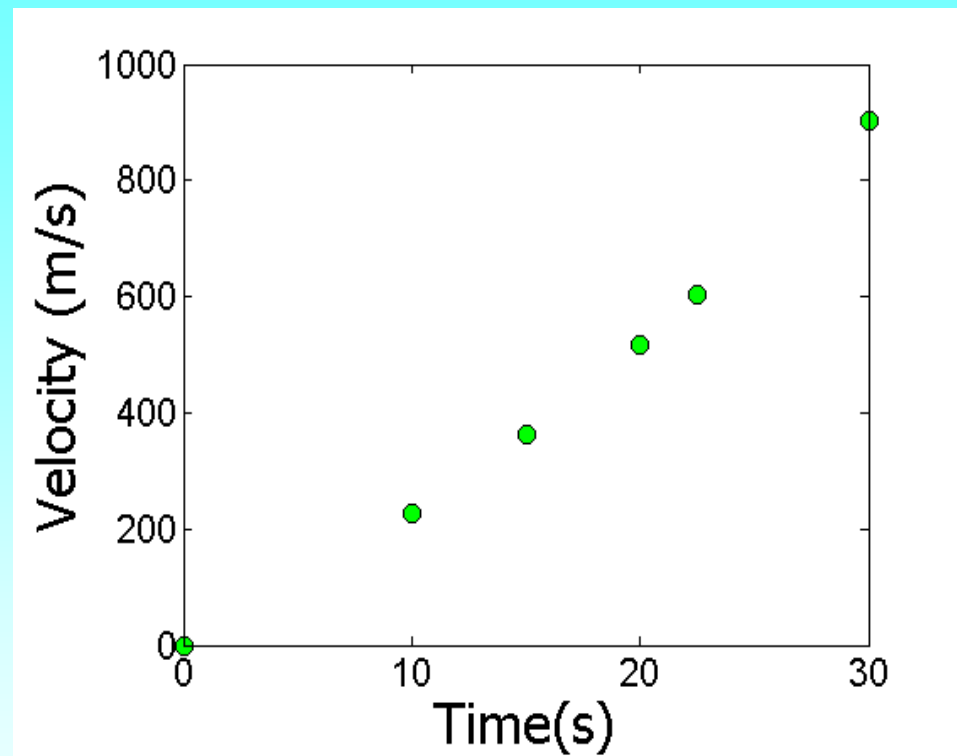
The upward velocity of a rocket is given as a function of time. Using quadratic splines

- a) Find the velocity at $t=16$ seconds
- b) Find the acceleration at $t=16$ seconds
- c) Find the distance covered between $t=11$ and $t=16$ seconds

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Data and Plot

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Solution

$$v(t) = a_1 t^2 + b_1 t + c_1, \quad 0 \leq t \leq 10$$

$$= a_2 t^2 + b_2 t + c_2, \quad 10 \leq t \leq 15$$

$$= a_3 t^2 + b_3 t + c_3, \quad 15 \leq t \leq 20$$

$$= a_4 t^2 + b_4 t + c_4, \quad 20 \leq t \leq 22.5$$

$$= a_5 t^2 + b_5 t + c_5, \quad 22.5 \leq t \leq 30$$

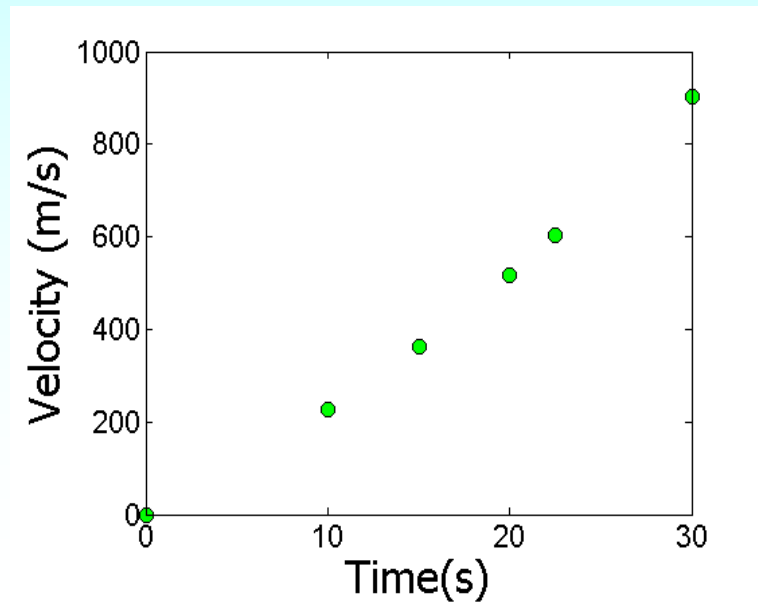
Let us set up the equations

Each Spline Goes Through Two Consecutive Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1, \quad 0 \leq t \leq 10$$

$$a_1(0)^2 + b_1(0) + c_1 = 0$$

$$a_1(10)^2 + b_1(10) + c_1 = 227.04$$



Each Spline Goes Through Two Consecutive Data Points

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$a_2(10)^2 + b_2(10) + c_2 = 227.04$$

$$a_2(15)^2 + b_2(15) + c_2 = 362.78$$

$$a_3(15)^2 + b_3(15) + c_3 = 362.78$$

$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$

$$a_4(20)^2 + b_4(20) + c_4 = 517.35$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97$$

$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

Derivatives are Continuous at Interior Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1, \quad 0 \leq t \leq 10$$

$$= a_2 t^2 + b_2 t + c_2, \quad 10 \leq t \leq 15$$

$$\left. \frac{d}{dt} (a_1 t^2 + b_1 t + c_1) \right|_{t=10} = \left. \frac{d}{dt} (a_2 t^2 + b_2 t + c_2) \right|_{t=10}$$

$$(2a_1 t + b_1) \Big|_{t=10} = (2a_2 t + b_2) \Big|_{t=10}$$

$$2a_1(10) + b_1 = 2a_2(10) + b_2$$

$$20a_1 + b_1 - 20a_2 - b_2 = 0$$

Derivatives are continuous at Interior Data Points

At $t=10$

$$2a_1(10) + b_1 - 2a_2(10) - b_2 = 0$$

At $t=15$

$$2a_2(15) + b_2 - 2a_3(15) - b_3 = 0$$

At $t=20$

$$2a_3(20) + b_3 - 2a_4(20) - b_4 = 0$$

At $t=22.5$

$$2a_4(22.5) + b_4 - 2a_5(22.5) - b_5 = 0$$

Last Equation

$$a_1 = 0$$

Final Set of Equations

$$\begin{bmatrix}
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\
 20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}
 \begin{bmatrix}
 a_1 \\
 b_1 \\
 c_1 \\
 a_2 \\
 b_2 \\
 c_2 \\
 a_3 \\
 b_3 \\
 c_3 \\
 a_4 \\
 b_4 \\
 c_4 \\
 a_5 \\
 b_5 \\
 c_5
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 227.04 \\
 227.04 \\
 362.78 \\
 362.78 \\
 517.35 \\
 517.35 \\
 602.97 \\
 602.97 \\
 901.67 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}$$

Coefficients of Spline

i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	-0.1356	35.66	-141.61
4	1.6048	-33.956	554.55
5	0.20889	28.86	-152.13

Final Solution

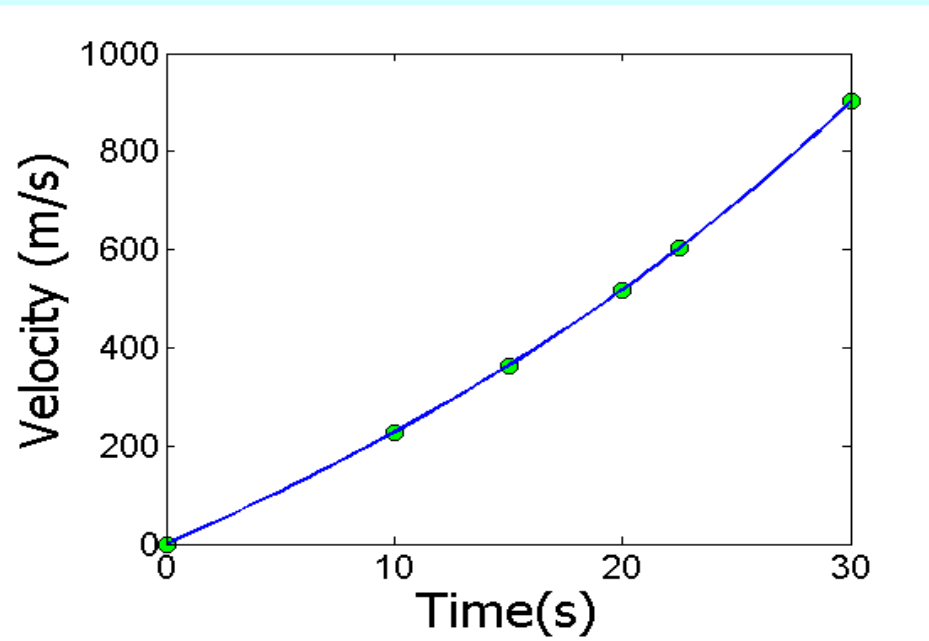
$$v(t) = 22.704t, \quad 0 \leq t \leq 10$$

$$= 0.8888t^2 + 4.928t + 88.88, \quad 10 \leq t \leq 15$$

$$= -0.1356t^2 + 35.66t - 141.61, \quad 15 \leq t \leq 20$$

$$= 1.6048t^2 - 33.956t + 554.55, \quad 20 \leq t \leq 22.5$$

$$= 0.20889t^2 + 28.86t - 152.13, \quad 22.5 \leq t \leq 30$$



Velocity at a Particular Point

a) Velocity at t=16

$$v(t) = 22.704t, \quad 0 \leq t \leq 10$$

$$= 0.8888t^2 + 4.928t + 88.88, \quad 10 \leq t \leq 15$$

$$= -0.1356t^2 + 35.66t - 141.61, \quad 15 \leq t \leq 20$$

$$= 1.6048t^2 - 33.956t + 554.55, \quad 20 \leq t \leq 22.5$$

$$= 0.20889t^2 + 28.86t - 152.13, \quad 22.5 \leq t \leq 30$$

$$\begin{aligned} v(16) &= -0.1356(16)^2 + 35.66(16) - 141.61 \\ &= 394.24 \text{ m/s} \end{aligned}$$

Acceleration from Velocity Profile

b) Acceleration at $t=16$

$$v(t) = 22.704t, \quad 0 \leq t \leq 10$$

$$= 0.8888t^2 + 4.928t + 88.88, \quad 10 \leq t \leq 15$$

$$= -0.1356t^2 + 35.66t - 141.61, \quad 15 \leq t \leq 20$$

$$= 1.6048t^2 - 33.956t + 554.55, \quad 20 \leq t \leq 22.5$$

$$= 0.20889t^2 + 28.86t - 152.13, \quad 22.5 \leq t \leq 30$$

$$a(16) = \left. \frac{d}{dt} v(t) \right|_{t=16}$$

Acceleration from Velocity Profile

The quadratic spline valid at $t=16$ is given by

$$v(t) = -0.1356t^2 + 35.66t - 141.61, \quad 15 \leq t \leq 20$$

$$a(t) = \frac{d}{dt}(-0.1356t^2 + 35.66t - 141.61)$$

$$= -0.2712t + 35.66, \quad 15 \leq t \leq 20$$

$$a(16) = -0.2712(16) + 35.66$$

$$= 31.321 \text{m/s}^2$$

Distance from Velocity Profile

c) Find the distance covered by the rocket from $t=11\text{s}$ to $t=16\text{s}$.

$$\begin{aligned}v(t) &= 22.704t, & 0 \leq t \leq 10 \\&= 0.8888t^2 + 4.928t + 88.88, & 10 \leq t \leq 15 \\&= -0.1356t^2 + 35.66t - 141.61, & 15 \leq t \leq 20 \\&= 1.6048t^2 - 33.956t + 554.55, & 20 \leq t \leq 22.5 \\&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \leq t \leq 30\end{aligned}$$

$$S(16) - S(11) = \int_{11}^{16} v(t) dt$$

Distance from Velocity Profile

$$v(t) = 0.8888t^2 + 4.928t + 88.88, \quad 10 \leq t \leq 15$$

$$= -0.1356t^2 + 35.66t - 141.61, \quad 15 \leq t \leq 20$$

$$\begin{aligned} S(16) - S(11) &= \int_{11}^{16} v(t)dt = \int_{11}^{15} v(t)dt + \int_{15}^{16} v(t)dt \\ &= \int_{11}^{15} (0.8888t^2 + 4.928t + 88.88)dt \\ &\quad + \int_{15}^{16} (-0.1356t^2 + 35.66t - 141.61)dt \\ &= 1595.9\text{m} \end{aligned}$$

END

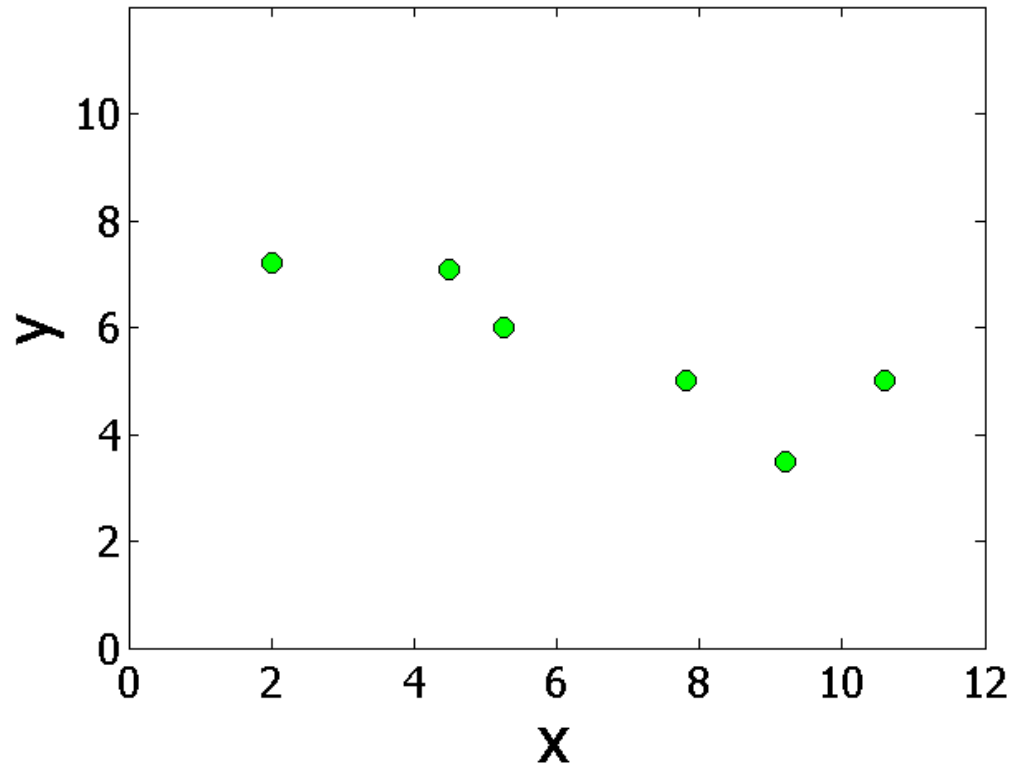
<http://numericalmethods.eng.usf.edu>

Find a Smooth Shortest Path for a Robot

<http://numericalmethods.eng.usf.edu>

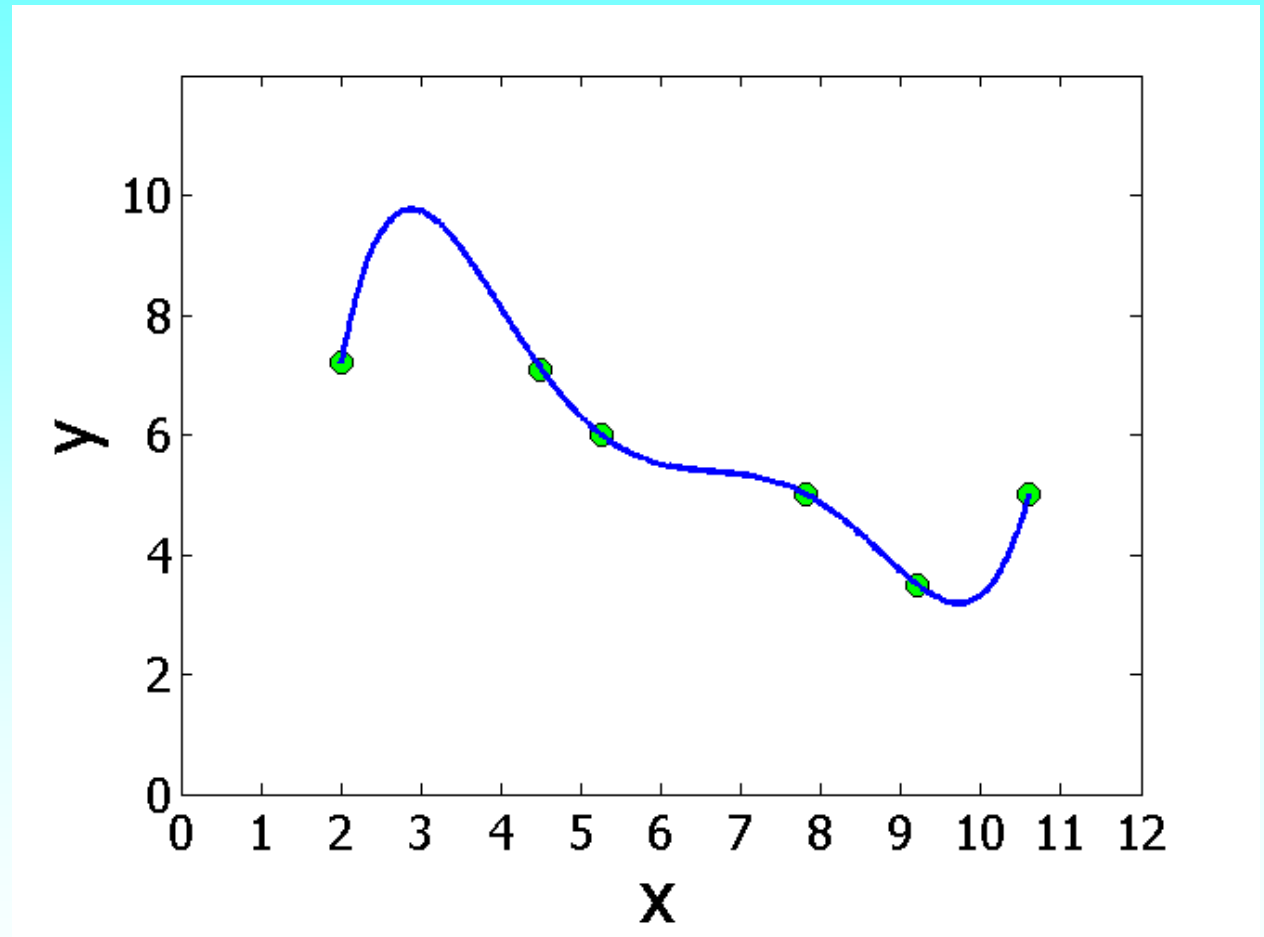
Points for Robot Path

x	y
2.00	7.2
4.5	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

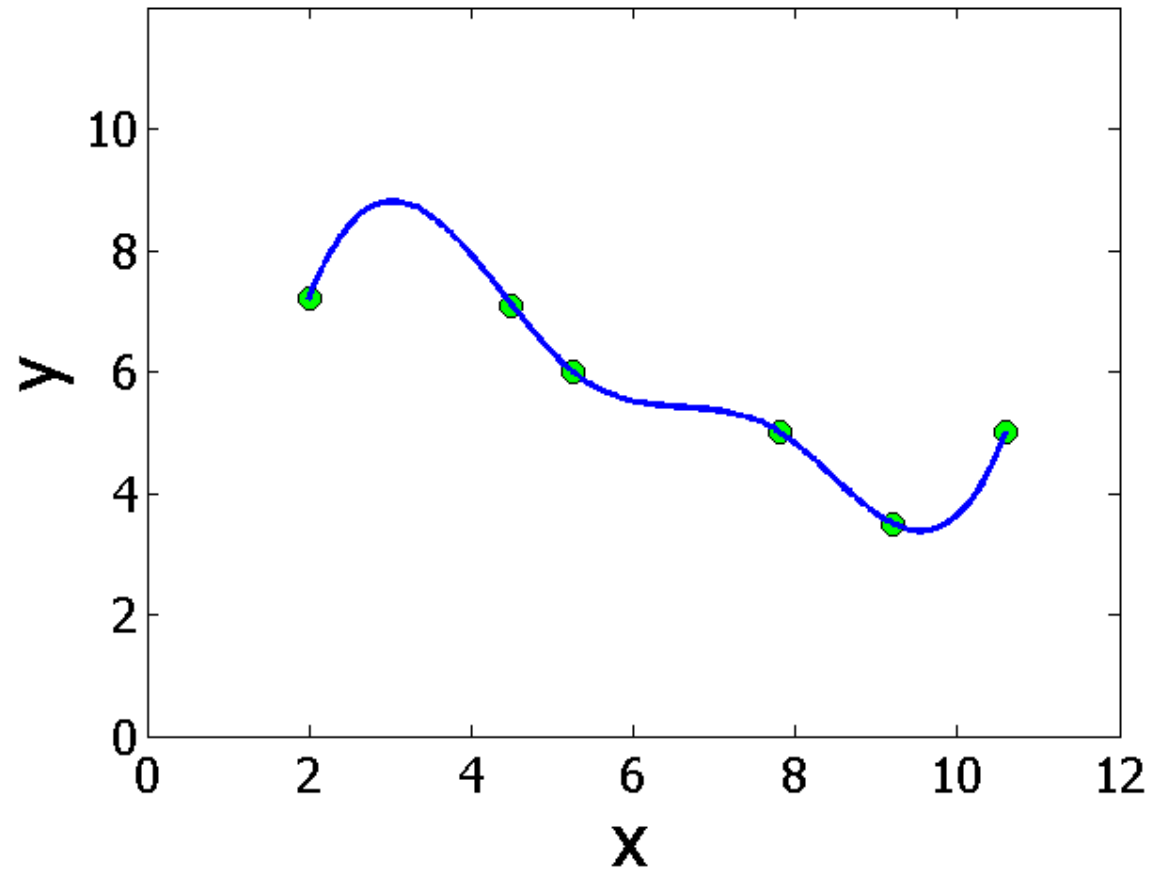


Find the shortest but smooth path through consecutive data points

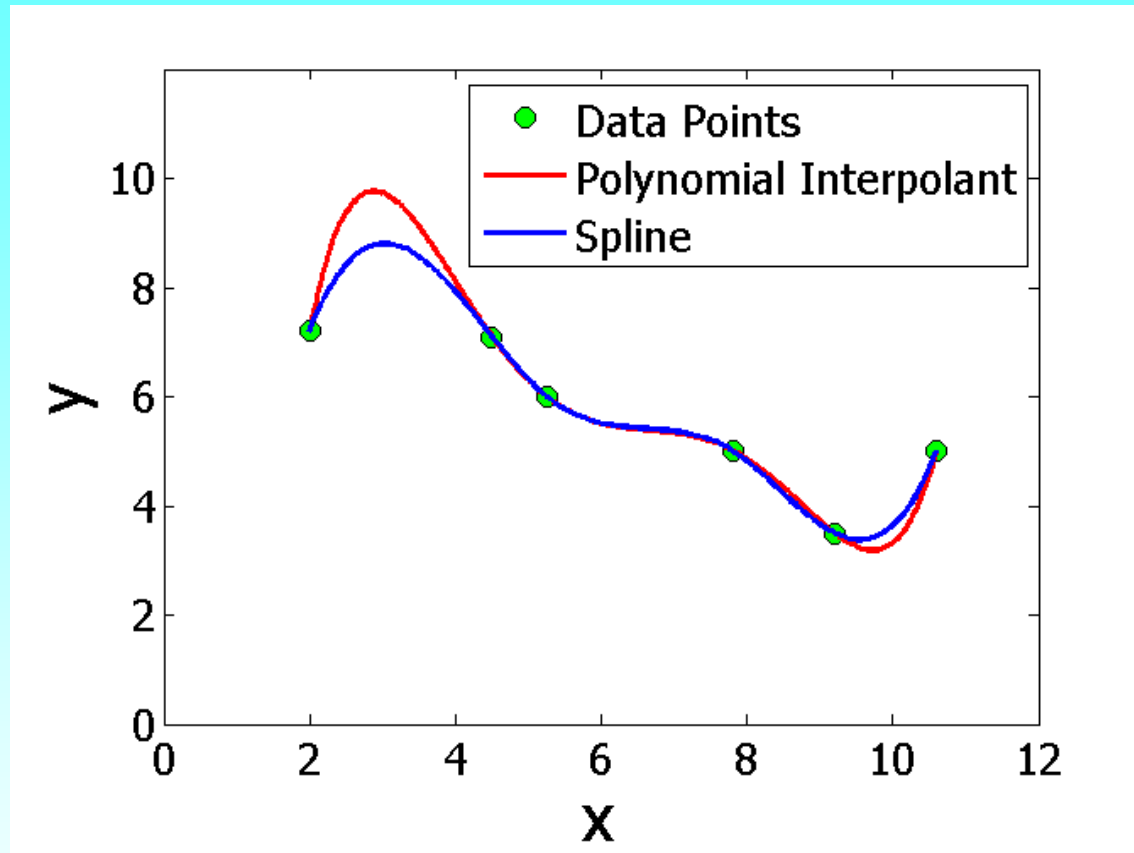
Polynomial Interpolant Path



Spline Interpolant Path



Compare Spline & Polynomial Interpolant Path



Length of path

Polynomial Interpolant=14.9

Spline Interpolant =12.9