Spline Interpolation Method

Major: All Engineering Majors

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Spline Method of Interpolation

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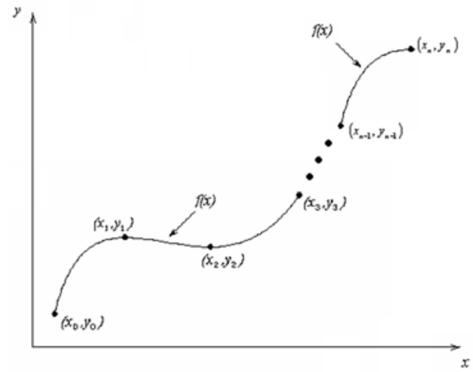
What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

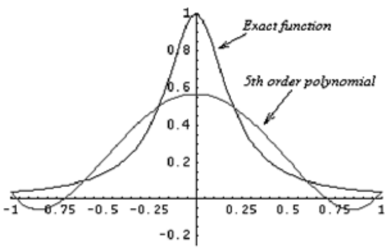


Figure : 5th order polynomial vs. exact function

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Why Splines ?

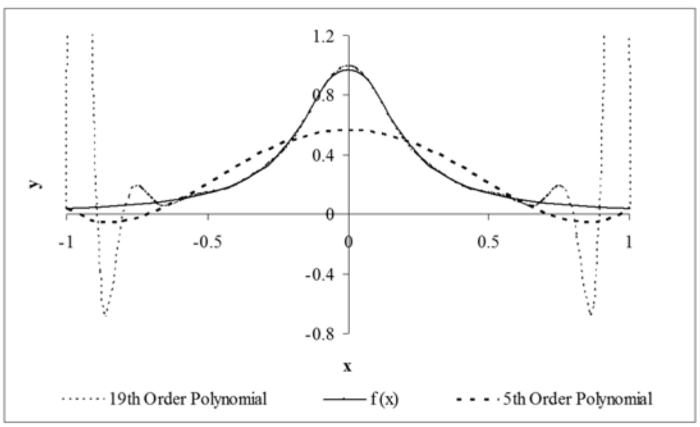

Figure : Higher order polynomial interpolation is a bad idea

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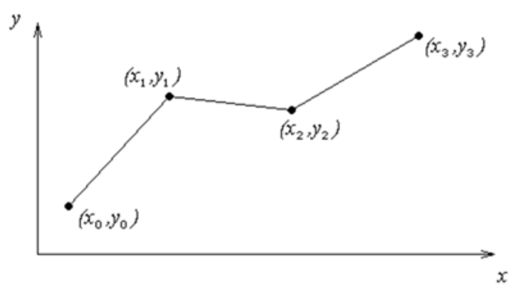
Linear Interpolation

Given $(x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $(y_i = f(x_i))$

Figure : Linear splines



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Linear Interpolation (contd)

$$
\begin{aligned}
f(x) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1 \\
&= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2 \\
&\quad . \\
&\quad . \\
&\quad . \\
&= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n
\end{aligned}
$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between x_{i-1} and x_i.

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Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using linear splines.

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

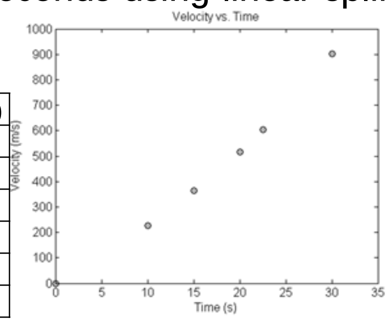

Figure. Velocity vs. time data for the rocket example

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Linear Interpolation

$$t_0 = 15, \qquad v(t_0) = 362.78$$

$$t_1 = 20, \qquad v(t_1) = 517.35$$

$$v(t) = v(t_0) + \frac{v(t_1) - v(t_0)}{t_1 - t_0}(t - t_0)$$

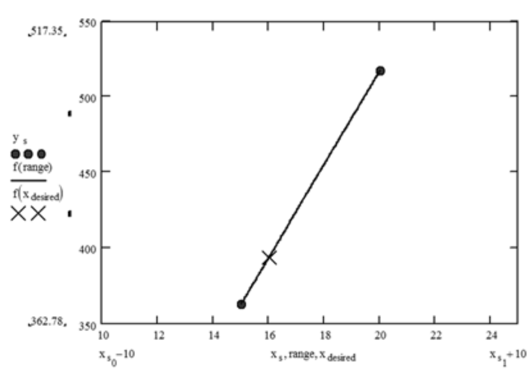
$$= 362.78 + \frac{517.35 - 362.78}{20 - 15}(t - 15)$$

$$v(t) = 362.78 + 30.913(t - 15)$$

At $t = 16$,

$$v(16) = 362.78 + 30.913(16 - 15)$$

$$= 393.7 \text{ m/s}$$



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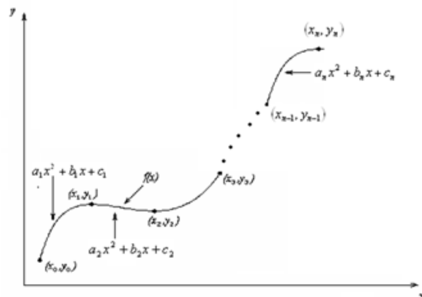
Quadratic Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, \qquad x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, \qquad x_1 \le x \le x_2 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= a_n x^2 + b_n x + c_n, \qquad x_{n-1} \le x \le x_n
\end{aligned}
$$



Find a_i, b_i, c_i, $i = 1, 2, \ldots, \text{n}$

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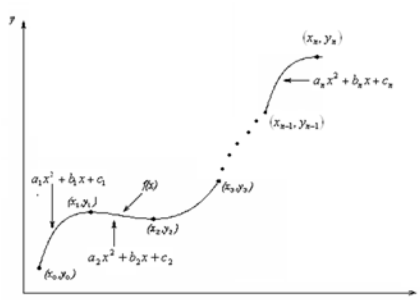
Quadratic Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$
\begin{aligned}
a_1 x_0^{\,2} + b_1 x_0 + c_1 &= f(x_0) \\
a_1 x_1^{\,2} + b_1 x_1 + c_1 &= f(x_1) \\
&. \\
&. \\
a_i x_{i-1}^{\,2} + b_i x_{i-1} + c_i &= f(x_{i-1}) \\
a_i x_i^{\,2} + b_i x_i + c_i &= f(x_i) \\
&. \\
&. \\
a_n x_{n-1}^{\,2} + b_n x_{n-1} + c_n &= f(x_{n-1}) \\
a_n x_n^{\,2} + b_n x_n + c_n &= f(x_n)
\end{aligned}
$$



This condition gives 2n equations

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Quadratic Splines (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$a_1x^2 + b_1x + c_1$ is $2a_1x + b_1$

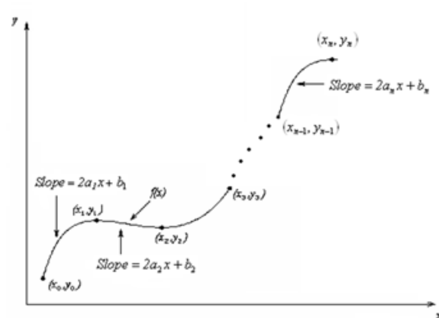
The derivative of the second spline

$a_2x^2 + b_2x + c_2$ is $2a_2x + b_2$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



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Quadratic Splines (contd)

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

.

.

.

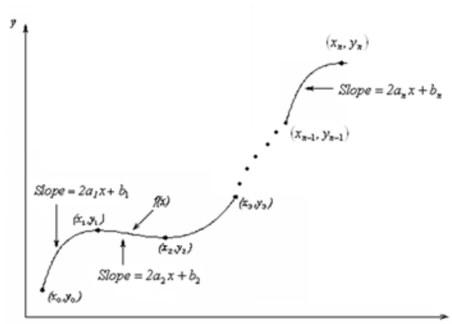
$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

.

.

.

$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



We have (n-1) such equations. The total number of equations is $(2n) + (n-1) = (3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

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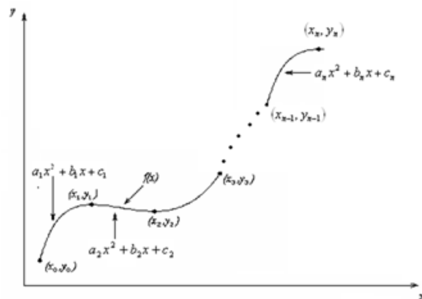
Quadratic Splines (contd)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

$$
\begin{aligned}
f(x) &= a_1x^2 + b_1x + c_1, \qquad && x_0 \le x \le x_1 \\
&= a_2x^2 + b_2x + c_2, && x_1 \le x \le x_2 \\
&. \\
&. \\
&. \\
&= a_nx^2 + b_nx + c_n, && x_{n-1} \le x \le x_n
\end{aligned}
$$



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Quadratic Spline Interpolation Part 1 of 2

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Quadratic Spline Example

The upward velocity of a rocket is given as a function of time. Using quadratic splines

a) Find the velocity at t=16 seconds

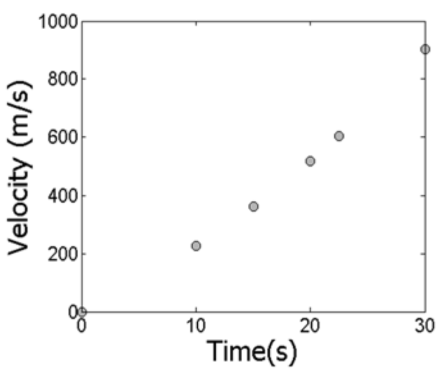
b) Find the acceleration at t=16 seconds

c) Find the distance covered between t=11 and t=16 seconds

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Data and Plot

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



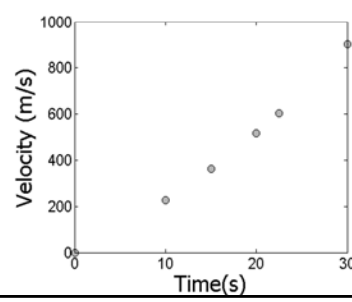
Solution

$$v(t) = a_1 t^2 + b_1 t + c_1, \quad 0 \le t \le 10$$
$$= a_2 t^2 + b_2 t + c_2, \quad 10 \le t \le 15$$
$$= a_3 t^2 + b_3 t + c_3, \quad 15 \le t \le 20$$
$$= a_4 t^2 + b_4 t + c_4, \quad 20 \le t \le 22.5$$
$$= a_5 t^2 + b_5 t + c_5, \quad 22.5 \le t \le 30$$

Let us set up the equations

Each Spline Goes Through Two Consecutive Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1, \; 0 \le t \le 10$$
$$a_1(0)^2 + b_1(0) + c_1 = 0$$
$$a_1(10)^2 + b_1(10) + c_1 = 227.04$$



Each Spline Goes Through Two Consecutive Data Points

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$a_2(10)^2 + b_2(10) + c_2 = 227.04$$
$$a_2(15)^2 + b_2(15) + c_2 = 362.78$$
$$a_3(15)^2 + b_3(15) + c_3 = 362.78$$
$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$
$$a_4(20)^2 + b_4(20) + c_4 = 517.35$$
$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97$$
$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97$$
$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

Derivatives are Continuous at Interior Data Points

$$v(t) = a_1t^2 + b_1t + c_1,\ 0 \le t \le 10$$
$$= a_2t^2 + b_2t + c_2, 10 \le t \le 15$$

$$\frac{d}{dt}\left(a_1t^2 + b_1t + c_1\right)\Big|_{t=10} = \frac{d}{dt}\left(a_2t^2 + b_2t + c_2\right)\Big|_{t=10}$$

$$\left(2a_1t + b_1\right)\Big|_{t=10} = \left(2a_2t + b_2\right)\Big|_{t=10}$$

$$2a_1(10) + b_1 = 2a_2(10) + b_2$$

$$20a_1 + b_1 - 20a_2 - b_2 = 0$$

Derivatives are continuous at Interior Data Points

At t=10

$$2a_1(10) + b_1 - 2a_2(10) - b_2 = 0$$

At t=15

$$2a_2(15) + b_2 - 2a_3(15) - b_3 = 0$$

At t=20

$$2a_3(20) + b_3 - 2a_4(20) - b_4 = 0$$

At t=22.5

$$2a_4(22.5) + b_4 - 2a_5(22.5) - b_5 = 0$$

Last Equation

$$a_1 = 0$$

Final Set of Equations

$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\
20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Coefficients of Spline

i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	-0.1356	35.66	-141.61
4	1.6048	-33.956	554.55
5	0.20889	28.86	-152.13

END

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Quadratic Spline Interpolation Part 2 of 2

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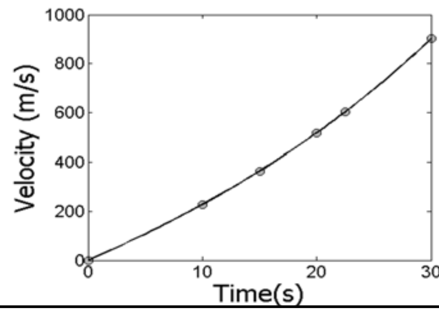
Final Solution

$$\begin{aligned} v(t) &= 22.704t, & 0 \le t \le 10 \\ &= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\ &= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\ &= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\ &= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30 \end{aligned}$$



Velocity at a Particular Point

a) Velocity at t=16

$$\begin{aligned} v(t) &= 22.704t, & 0 \le t \le 10 \\ &= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\ &= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\ &= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\ &= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30 \end{aligned}$$

$$\begin{aligned} v(16) &= -0.1356(16)^2 + 35.66(16) - 141.61 \\ &= 394.24 \text{ m/s} \end{aligned}$$

Acceleration from Velocity Profile

b) Acceleration at t=16

$$
\begin{aligned}
v(t) &= 22.704t, & 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30
\end{aligned}
$$

$$a(16) = \frac{d}{dt} v(t)\Big|_{t=16}$$

Acceleration from Velocity Profile

The quadratic spline valid at t=16 is given by

$$v(t) = -0.1356\,t^2 + 35.66t - 141.61, \quad 15 \le t \le 20$$

$$a(t) = \frac{d}{dt}(-0.1356t^2 + 35.66t - 141.61)$$

$$= -0.2712t + 35.66, \quad 15 \le t \le 20$$

$$a(16) = -0.2712(16) + 35.66 = 31.321\text{m/s}^2$$

Distance from Velocity Profile

c) Find the distance covered by the rocket from t=11s to t=16s.

$$
\begin{aligned}
v(t) &= 22.704t, && 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, && 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, && 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, && 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, && 22.5 \le t \le 30
\end{aligned}
$$

$$S(16) - S(11) = \int_{11}^{16} v(t)dt$$

Distance from Velocity Profile

$$
\begin{aligned}
v(t) &= 0.8888t^2 + 4.928t + 88.88, \quad 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, \quad 15 \le t \le 20
\end{aligned}
$$

$$
\begin{aligned}
S(16) - S(11) &= \int_{11}^{16} v(t)dt = \int_{11}^{15} v(t)dt + \int_{15}^{16} v(t)dt \\
&= \int_{11}^{15} (0.8888t^2 + 4.928t + 88.88)dt \\
&\quad + \int_{15}^{16} (-0.1356t^2 + 35.66t - 141.61)dt \\
&= 1595.9 \text{ m}
\end{aligned}
$$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/spline_method.html

END

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Find a Smooth Shortest Path for a Robot

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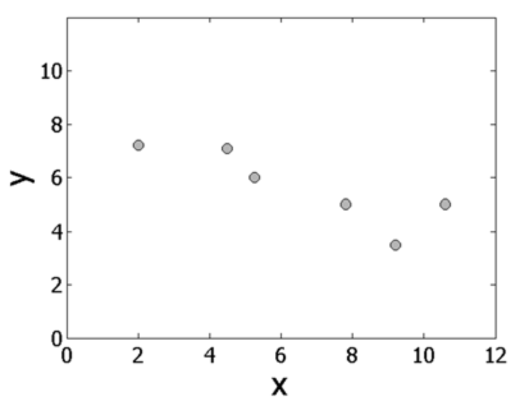
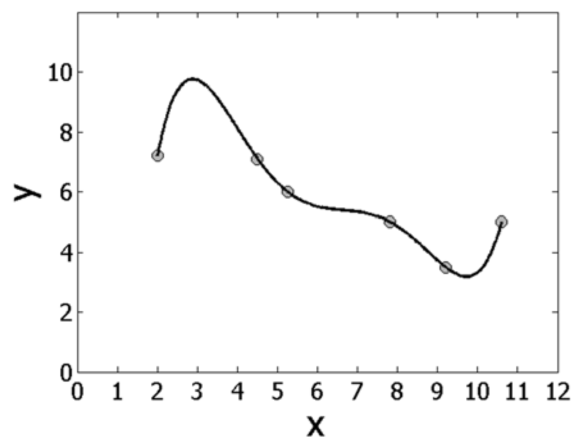
Points for Robot Path

x	*y*
2.00	7.2
4.5	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

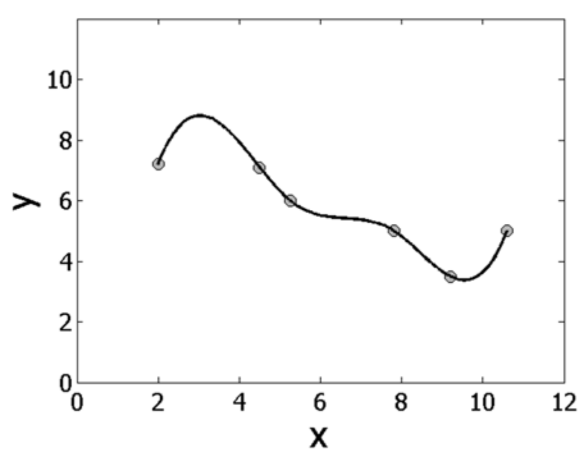


Find the shortest but smooth path through consecutive data points

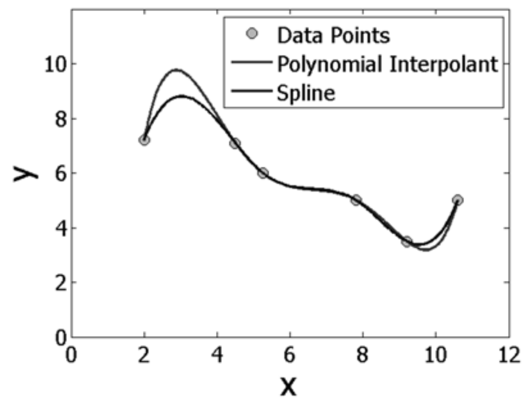
Polynomial Interpolant Path



Spline Interpolant Path



Compare Spline & Polynomial Interpolant Path



Length of path

Polynomial Interpolant=14.9

Spline Interpolant =12.9

THE END