Two-Point Gaussian Quadrature Rule

Background

Trapezoidal Rule by undetermined coefficients

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

Basis of the Two-Point Gaussian Quadrature Rule

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

How do I find the 4 unknowns?

The four unknowns x_1, x_2, c_1 and c_2 are found by assuming that the formula

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

gives exact results for integrating a general third order polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Exact integral is

$$\int_a^b f(x)dx = \int_a^b (a_0 + a_1 x + a_2 x^2 + a_3 x^3)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

This is what I get from the exact integration

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$$= a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right)$$

This is what I get from the RHS!

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$$\int_a^b f(x)dx = c_1(a_0 + a_1 x_1 + a_2 {x_1}^2 + a_3 {x_1}^3)$$

$$+c_2(a_0 + a_1 x_2 + a_2 {x_2}^2 + a_3 {x_2}^3)$$

$$= a_0 (c_1 + c_2)$$
$$+a_1(c_1 x_1 + c_2 x_2)$$
$$+a_2(c_1 {x_1}^2 + c_2 {x_2}^2)$$
$$+a_3(c_1 {x_1}^3 + c_2 {x_2}^3)$$

Equating exact value with formula

Equating equations

$$a_0(b-a)+a_1\left(\frac{b^2-a^2}{2}\right)+a_2\left(\frac{b^3-a^3}{3}\right)+a_3\left(\frac{b^4-a^4}{4}\right)$$

$$=a_0(c_1+c_2)+a_1(c_1x_1+c_2x_2)+a_2(c_1{x_1}^2+c_2{x_2}^2)+a_3(c_1{x_1}^3+c_2{x_2}^3)$$

So we get 4 simultaneous NLEs

Since the constants a_0, a_1, a_2, a_3 are arbitrary

$$c_1 + c_2 = b - a$$

$$c_1 x_1 + c_2 x_2 = \frac{b^2 - a^2}{2}$$

$$c_1 {x_1}^2 + c_2 {x_2}^2 = \frac{b^3 - a^3}{3}$$

$$c_1 {x_1}^3 + c_2 {x_2}^3 = \frac{b^4 - a^4}{4}$$

Solution of 4 simultaneous NLEs

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$$= \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)$$

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