

Two-Point Gaussian Quadrature Rule

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Background

Trapezoidal Rule by undetermined coefficients

$$\begin{aligned}\int_a^b f(x)dx &\approx c_1 f(a) + c_2 f(b) \\ &= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)\end{aligned}$$

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Basis of the Two-Point Gaussian Quadrature Rule

$$\begin{aligned}\int_a^b f(x)dx &\approx c_1 f(a) + c_2 f(b) \\ \int_a^b f(x)dx &\approx c_1 f(x_1) + c_2 f(x_2)\end{aligned}$$

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How do I find the 4 unknowns?

The four unknowns x_1 , x_2 , c_1 and c_2 are found by assuming that the formula

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

gives exact results for integrating a general third order polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Exact integral is

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^b (a_0 + a_1 x + a_2 x^2 + a_3 x^3)dx \\ &= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} \right]_a^b \\ &= a_0(b-a) + a_1 \left(\frac{b^2 - a^2}{2} \right) + a_2 \left(\frac{b^3 - a^3}{3} \right) + a_3 \left(\frac{b^4 - a^4}{4} \right)\end{aligned}$$

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This is what I get from the formula

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3.$$

$$\begin{aligned} c_1f(x_1) + c_2f(x_2) &= c_1(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3) \\ &\quad + c_2(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3) \\ &= a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) \\ &\quad + a_2(c_1x_1^2 + c_2x_2^2) + a_3(c_1x_1^3 + c_2x_2^3) \end{aligned}$$

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Equating exact value with formula

$$\begin{aligned} &a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right) \\ &= a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2(c_1x_1^2 + c_2x_2^2) + a_3(c_1x_1^3 + c_2x_2^3) \end{aligned}$$

Since the constants a_0, a_1, a_2, a_3 are arbitrary

$$c_1 + c_2 = b - a$$

$$c_1x_1 + c_2x_2 = \frac{b^2 - a^2}{2}$$

$$c_1x_1^2 + c_2x_2^2 = \frac{b^3 - a^3}{3}$$

$$c_1x_1^3 + c_2x_2^3 = \frac{b^4 - a^4}{4}$$

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Solution of 4 simultaneous NLEs

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

$$x_1 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

$$\int_a^b f(x)dx \approx c_1f(x_1) + c_2f(x_2)$$

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Final Expression for 2-point
Gauss Quadrature Rule

$$\begin{aligned} \int_a^b f(x)dx &\approx c_1f(x_1) + c_2f(x_2) \\ &= \frac{b-a}{2}f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) \\ &\quad + \frac{b-a}{2}f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) \end{aligned}$$

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