Adequacy of Solutions

Autar Kaw
Humberto Isaza

"the magnitude of the determinant is an indication of neither the condition of the matrix nor the accuracy of the solution" - Henry Thacher

http://nm.MathForCollege.com
Transforming Numerical Methods Education for STEM Undergraduates

Norm of a matrix

$If\ [A]\ is\ a\ m \times n\ matrix, \|A\|_\infty = \max\left(\sum_{j=1}^{n}|a_{ij}|\right), 1 \leq i \leq m$

Find the row sum norm of the following matrix $A = \begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix}$

$$\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n}|a_{ij}|$$

$$= \max[(|10| + |-7| + |0|), (|-3| + |2.099| + |6|), (|5| + |-1| + |5|)]$$

$$= \max[(10 + 7 + 0), (3 + 2.099 + 6), (5 + 1 + 5)]$$

$$= \max[17, 11.099, 11]$$

$$= 17.$$

Accuracy of Solution

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using 6 significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.9625 \\ 1.05 \\ 0.999995 \end{bmatrix}$$

Accuracy of Solution

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using 5 significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 1.5 \\ 0.99995 \end{bmatrix}$$

Well-conditioned and ill-conditioned

What do you mean by ill-conditioned and well-conditioned system of equations?

A system of equations is considered to be **well-conditioned** if a small change in the coefficient matrix or a small change in the right hand side results in a small change in the solution vector.

A system of equations is considered to be **ill-conditioned** if a small change in the coefficient matrix or a small change in the right hand side results in a large change in the solution vector.

Is this system of equations well-conditioned or ill-conditioned?

$$\begin{bmatrix}1 & 2\\ 2 & 3.999\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ 7.999\end{bmatrix} \qquad \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2\\ 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\ 2 & 3.999\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4.001\\ 7.998\end{bmatrix} \qquad \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-3.999\\ 4.000\end{bmatrix}$$

$$\begin{bmatrix}1.001 & 2.001\\ 2.001 & 3.998\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ 7.999\end{bmatrix} \qquad \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}3.994\\ 0.001388\end{bmatrix}$$

Is this system of equations well-conditioned or ill-conditioned?

$$\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ 7\end{bmatrix} \qquad \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2\\ 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4.001\\ 7.001\end{bmatrix} \qquad \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1.999\\ 1.001\end{bmatrix}$$

$$\begin{bmatrix}1.001 & 2.001\\ 2.001 & 3.001\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ 7\end{bmatrix} \qquad \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2.003\\ 0.997\end{bmatrix}$$

How does norm help in finding conditioning of the matrix?

$$\begin{bmatrix}1 & 2\\ 2 & 3.999\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ 7.999\end{bmatrix} \qquad \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2\\ 1\end{bmatrix} \qquad \|C\|_\infty = 7.999 \qquad \|X\|_\infty = 2$$

$$\begin{bmatrix}1 & 2\\ 2 & 3.999\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4.001\\ 7.998\end{bmatrix} \qquad \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-3.999\\ 4.000\end{bmatrix}$$

$$[\Delta C]=\begin{bmatrix}4.001\\ 7.998\end{bmatrix}-\begin{bmatrix}4\\ 7.999\end{bmatrix}=\begin{bmatrix}0.001\\ -0.001\end{bmatrix} \qquad \|\Delta C\|_\infty = 0.001$$

$$[\Delta X]=\begin{bmatrix}-3.999\\ 4.000\end{bmatrix}-\begin{bmatrix}2\\ 1\end{bmatrix}=\begin{bmatrix}-5.999\\ 3.000\end{bmatrix} \qquad \|\Delta X\|_\infty = 5.999$$

$$\frac{\|\Delta C\|_\infty}{\|C\|_\infty}=\frac{0.001}{7.999}=1.250\times 10^{-4} \qquad \frac{\|\Delta X\|_\infty}{\|X\|_\infty}=\frac{5.999}{2}=2.9995$$

How does norm help in finding conditioning of the matrix?

$$\frac{\|\Delta X\|_\infty / \|X\|_\infty}{\|\Delta C\|_\infty / \|C\|_\infty} = \frac{2.9995}{1.250 \times 10^{-4}}$$

$$= 23993$$

Identifying well-conditioned and ill conditioned system of equations

There is a relationship that exists between

$$\frac{\|\Delta X\|}{\|X\|} \text{ and } \frac{\|\Delta C\|}{\|C\|}$$

and between

$$\frac{\|\Delta X\|}{\|X\|} \text{ and } \frac{\|\Delta A\|}{\|A\|}$$

These relationships are

$$\frac{\|\Delta X\|}{\|X + \Delta X\|} \leq \|A\| \, \|A^{-1}\| \frac{\|\Delta C\|}{\|C\|} \qquad \frac{\|\Delta X\|}{\|X\|} \leq \|A\| \, \|A^{-1}\| \frac{\|\Delta A\|}{\|A\|}$$

How many sig digits can I trust in my solution?

$$\begin{bmatrix} 1 & 2 \\ 2 & 3.999 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

How many significant digits can I trust in the solution of the above system of equations?

$$[A] = \begin{bmatrix} 1 & 2 \\ 2 & 3.999 \end{bmatrix} \qquad [A]^{-1} = \begin{bmatrix} -3999 & 2000 \\ 2000 & -1000 \end{bmatrix}$$

$$\|A\|_\infty = 5.999$$

$$\|A^{-1}\|_\infty = 5999$$

$$Cond(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

$$= 5.999 \times 5999.4$$

$$= 35990$$

How many sig digits can I trust in my solution? (cont)

Assuming single precision with 23 bits used for the magnitude of mantissa for real numbers, the machine epsilon is

$$\epsilon_{mach} = 2^{-23} = 0.119209 \times 10^{-6}$$

$$Cond(A) \times \epsilon_{mach} = 35990 \times 0.119209 \times 10^{-6} = 0.4290 \times 10^{-2}$$

Comparing it with 0.5×10^{-m}

$$0.4290 \times 10^{-2} \leq 0.5 \times 10^{-m}$$

So two significant digits are at least correct in the solution vector as

$$m \leq 2$$