LU Decomposition Method

Major: All Engineering Majors

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LU Decomposition Method

[A][X] = [C]

1. Decompose [A] into [L] and [U]
2. Solve [L][Z] = [C] for [Z]
3. Solve [U][X] = [Z] for [X]

THE END

Is LU Decomposition better than Gaussian Elimination?

Solve $[A][X] = [B]$

T = clock cycle time and $n \times n$ = size of the matrix

Forward Elimination

$$CT|_{FE} = T\left(\frac{8n^3}{3} + 8n^2 - \frac{32n}{3}\right)$$

Decomposition to LU

$$CT|_{DE} = T\left(\frac{8n^3}{3} + 4n^2 - \frac{20n}{3}\right)$$

Back Substitution

$$CT|_{BS} = T(4n^2 + 12n)$$

Forward Substitution

$$CT|_{FS} = T(4n^2 - 4n)$$

Back Substitution

$$CT|_{BS} = T(4n^2 + 12n)$$

Is LU Decomposition better than Gaussian Elimination?

To solve $[A][X] = [B]$

Time taken by methods

Gaussian Elimination	LU Decomposition
$T\left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3}\right)$	$T\left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3}\right)$

T = clock cycle time and $n \times n$ = size of the matrix

So both methods are equally efficient.

THE END

Time Taken by Back Substitution

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \ldots, 1$$

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n = length(B);
X(n)=B(n)/A(n,n)
for i=n-1:-1:1
X(i) = B(i);
    for j=(i+1):1:n
        X(i) = X(i) - A(i, j)*X(j);
    end
X(i) = X(i)/A(i, i);
end
```

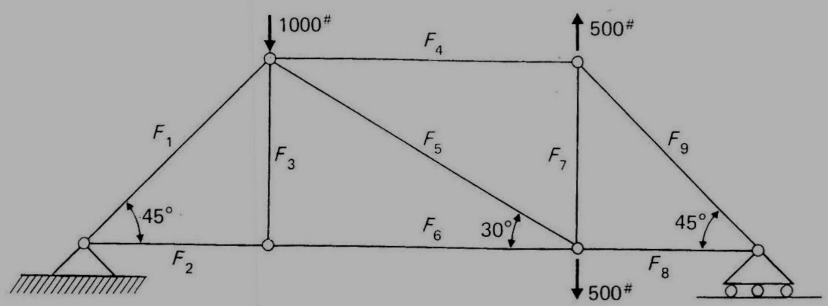
Back Substitution CT

$$CT|_{BS} = T(4n^2 + 12n)$$

THE END

Truss Problem

$$\begin{bmatrix} 0.7071 & 0 & 0 & -1 & -0.8660 & 0 & 0 & 0 & 0 \\ 0.7071 & 0 & 1 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.7071 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -0.7071 \\ 0 & 0 & 0 & 0 & 0.8660 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.7071 \end{bmatrix} F = \begin{bmatrix} 0 \\ -1000 \\ 0 \\ 0 \\ 500 \\ 0 \\ 0 \\ -500 \\ 0 \end{bmatrix}$$



Finding the inverse of a square matrix

The inverse [B] of a square matrix [*A*] is defined as

$$[A][B] = [I] \quad \textbf{OR} \quad [B][A] = [I]$$

Finding the inverse of a square matrix

How can LU Decomposition be used to find the inverse?

$$[A][B] = [I]$$

First Column of [B]

$$[A]\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second Column of [B] ....

$$[A]\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Last Column of [B]

$$[A]\begin{bmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Example: Inverse of a Matrix

Find the inverse of a square matrix [*A*]

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Using the decomposition procedure, the [*L*] and [*U*] matrices are found to be

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Solving for each column of [*B*] requires two steps

1) Solve [*L*] [*Z*] = [*C*] for [*Z*]
2) Solve [*U*] [*X*] = [*Z*] for [*X*]

Example: Inverse of a Matrix

$$\textit{Step 1: } [L][Z] = [C] \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} z_1 &= 1 \\ 2.56z_1 + z_2 &= 0 \\ 5.76z_1 + 3.5z_2 + z_3 &= 0 \end{aligned}$$

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$\begin{aligned} z_1 &= 1 \\ z_2 &= 0 - 2.56z_1 \\ &= 0 - 2.56(1) \\ &= -2.56 \\ z_3 &= 0 - 5.76z_1 - 3.5z_2 \\ &= 0 - 5.76(1) - 3.5(-2.56) \\ &= 3.2 \end{aligned}$$

Example: Inverse of a Matrix

$$Step\ 2\text{: } [U][X] = [Z] \rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$25b_{11} + 5b_{21} + b_{31} = 1$$
$$-4.8b_{21} - 1.56b_{31} = -2.56$$
$$0.7b_{31} = 3.2$$

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560b_{31}}{-4.8} = \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25} = \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762$$

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Example: Inverse of a Matrix

First Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Example: Inverse of a Matrix

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix} \qquad \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix} \qquad \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

To check your work do the following operation

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

To find inverse of [A]

Time taken by Gaussian Elimination

$$= n(CT|_{FE} + CT|_{BS})$$

$$= n \times T\left(\frac{8n^3}{3} + 8n^2 - \frac{32n}{3}\right) + n \times T(4n^2 + 12n)$$

$$= T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

Time taken by LU Decomposition

$$= CT|_{DE} + n \times CT|_{FS} + n \times CT|_{BS}$$

$$= T\left(\frac{8n^3}{3} + 4n^2 - \frac{20n}{3}\right) + n \times T(4n^2 - 4n) + n \times T(4n^2 + 12n)$$

$$= T\left(\frac{32n^3}{3} + 12n^2 - \frac{20n}{3}\right)$$

To find inverse of [A]

Time taken by Gaussian Elimination

$$T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

Time taken by LU Decomposition

$$T\left(\frac{32n^3}{3} + 12n^2 - \frac{20n}{3}\right)$$

Table 1 Comparing computational times of finding inverse of a matrix using LU decomposition and Gaussian elimination.

n	10	100	1000	10000
$\dfrac{CT_{\text{inverse GE}}}{CT_{\text{inverse LU}}}$	3.288	25.84	250.8	2501

For large n, $\dfrac{CT_{\text{inverse GE}}}{CT_{\text{inverse LU}}} \approx \dfrac{n}{4}$

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