Example

Write the following ordinary differentiation equation in the state variable form as a matrix

$$17\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 5y = 11e^{-t},$$

$$y(0) = 13,\ \ \frac{dy}{dt}(0) = 19,\ \ \frac{d^2y}{dt^2} = 23$$

Solution

The differential equation is of the third order, and we will have three state variables. Let these be named as

$$x_1 = y \qquad (1)$$

$$x_2 = \dot{y} \qquad (2)$$

$$x_3 = \ddot{y} \qquad (3)$$

Note the symbols. The symbol $\dot{y}$ stands for $\frac{dy}{dt}$ and $\ddot{y}$ stands for $\frac{d^2y}{dt^2}$. We are using these symbols as they are the norm in most textbooks.

We also have

$$\dot{x_3} = \dddot{y} \qquad (4)$$

So the given ordinary differential equation

$$17\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 5y = 11e^{-t},$$

can be written as

$$17\dot{x_3} + 3x_3 + 7x_2 + 5x_1 = 11e^{-t} \qquad (5)$$

Writing equations (2), (3) and (5) with first derivatives on the left side, we get

$$\dot{x_1} = \dot{y} = x_2$$

$$\dot{x_2} = \ddot{y} = x_3$$

$$\dot{x_3} = \frac{11e^{-t} - 3x_3 - 7x_2 - 5x_1}{17}$$

Rewriting the above three equations with coefficients for all the state variables, we get

$$\dot{x_1} = 0x_1 + 1x_2 + 0x_3 + 0$$

$$\dot{x_2} = 0x_1 + 0x_2 + 1x_3 + 0$$

$$\dot{x_3} = -\frac{5}{17}x_1 - \frac{7}{17}x_2 - \frac{3}{17}x_3 + \frac{11e^{-t}}{17}$$

In the matrix form

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{5}{17} & -\frac{7}{17} & -\frac{3}{17} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{11e^{-t}}{17} \end{bmatrix}$$

where the conditions are given by

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ \dot{y}(0) \\ \ddot{y}(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ \frac{dy}{dt}(0) \\ \frac{d^2y}{dt^2}(0) \end{bmatrix} = \begin{bmatrix} 13 \\ 19 \\ 23 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$