Chapter 2 - Macromechanical Analysis of a Lamina

Exercise Set

2.1 The number of independent elastic constants in three dimensions are:

Anisotropic	21
Monoclinic	13
Orthotropic	9
Transversely Orthotropic	5
Isotropic	2

2.2

$$\mathbf{C} = \begin{bmatrix} 13.675 & 6.39 & 10.846 & 0 & 0 & 0 \\ 6.39 & 6.58 & 6.553 & 0 & 0 & 0 \\ 10.846 & 6.553 & 12.316 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \cdot \text{Msi}$$

$$\mathbf{S} = \begin{bmatrix} 0.25 & -0.05 & -0.1935 & 0 & 0 & 0 \\ -0.05 & 0.3333 & -0.1333 & 0 & 0 & 0 \\ -0.1935 & -0.1333 & 0.3226 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1429 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1667 \end{bmatrix} \cdot \frac{1}{\text{Msi}}$$

2.3

a)

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} -8.269 \\ -6.731 \\ -4.423 \\ 0 \\ 10 \\ 9 \end{bmatrix} \cdot \text{kPa}$$

b) Compliance matrix -

$$\mathbf{S} = \begin{bmatrix} 0.5 & -0.5 & -0.4 & 0 & 0 & 0 \\ -0.5 & 0.3 & -0.2 & 0 & 0 & 0 \\ -0.4 & -0.2 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.667 \end{bmatrix} \cdot \frac{1}{GPa}$$

c)

$E_1 = 2 \cdot GPa$

$E_2 = 3.333 \cdot GPa$

$E_3 = 1.667 \cdot GPa$

$\nu_{12} = 1$

$\nu_{23} = 0.6667$

$\nu_{31} = 0.6667$

$G_{12} = 4 \cdot GPa$

$G_{23} = 2 \cdot GPa$

$G_{31} = 1.5 \cdot GPa$

d)

$W = 3.335 \cdot 10^{-2} \cdot Pa$

2.10

$$\mathbf{S} = \begin{pmatrix} 4.902 \cdot 10^{-3} & -1.127 \cdot 10^{-3} & 0 \\ -1.127 \cdot 10^{-3} & 5.405 \cdot 10^{-2} & 0 \\ 0 & 0 & 1.789 \cdot 10^{-1} \end{pmatrix} \cdot \frac{1}{GPa}$$

$$\mathbf{Q} = \begin{pmatrix} 204.98 & 4.28 & 0 \\ 4.28 & 18.59 & 0 \\ 0 & 0 & 5.59 \end{pmatrix} \cdot GPa$$

2.11

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} 17.35 \\ 103.59 \\ -536.70 \end{pmatrix} \cdot \frac{\mu m}{m}$$

2.12 Modifying Equation (2.17) for an isotropic lamina under plane stress -

$$\mathbf{S}=\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix}$$

Inverting the compliance matrix gives the reduced stiffness matrix -

$$\mathbf{Q}=\begin{bmatrix} \frac{E}{1-\nu^2} & \frac{E\cdot\nu}{1-\nu^2} & 0 \\ \frac{E\cdot\nu}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix}$$

2.13 Compliance matrix for a two dimensional orthotropic material per Equations (2.87) -

$$\mathbf{S}=\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$

Matrix inversion yields -

$$\mathbf{Q}=\begin{bmatrix} \frac{1}{1-\nu_{21}\cdot\nu_{12}}\cdot E_1 & \frac{\nu_{12}}{1-\nu_{21}\cdot\nu_{12}}\cdot E_2 & 0 \\ \frac{\nu_{21}}{1-\nu_{21}\cdot\nu_{12}}\cdot E_1 & \frac{1}{1-\nu_{21}\cdot\nu_{12}}\cdot E_2 & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

Compliance matrix for three dimensional orthotropic material per Equation (2.70) -

$$
\mathbf{S}=\begin{bmatrix}
\frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0\\
-\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0\\
-\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0\\
0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}}
\end{bmatrix}
$$

Matrix inversion yields -

$$
C_{11}=\frac{\left(1-\nu_{32}\cdot\nu_{23}\right)}{\left(1-\nu_{32}\cdot\nu_{23}-\nu_{21}\cdot\nu_{12}-\nu_{21}\cdot\nu_{32}\cdot\nu_{13}-\nu_{31}\cdot\nu_{12}\cdot\nu_{23}-\nu_{31}\cdot\nu_{13}\right)}\cdot E_1
$$

$$
C_{66}=G_{12}
$$

Proving $Q_{11} \neq C_{11}$ and $Q_{66} = C_{66}$.

2.15

$E_1 = 5.599 \cdot \mathrm{Msi}$, $E_2 = 1.199 \cdot \mathrm{Msi}$ $\nu_{12} = 0.2600$

$G_{12} = 0.6006 \cdot \mathrm{Msi}$

2.16

$$
\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{pmatrix}=\begin{pmatrix}9.019\cdot 10^{-1}\\ 5.098\\ -2.366\end{pmatrix}\cdot \mathrm{MPa}
$$

2.17

$$
\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}=\begin{pmatrix}2.201\\ 3.799\\ -3.232\end{pmatrix}\cdot\frac{\mu\mathrm{in}}{\mathrm{in}}
$$

2.18

$$\underline{Q} = \begin{pmatrix} 29.06 & 40.40 & 19.50 \\ 40.40 & 122.26 & 61.21 \\ 19.50 & 61.21 & 41.71 \end{pmatrix} \cdot GPa$$

$$\underline{S} = \begin{pmatrix} 6.383 \cdot 10^{-2} & -2.319 \cdot 10^{-2} & 4.195 \cdot 10^{-3} \\ -2.319 \cdot 10^{-2} & 3.925 \cdot 10^{-2} & -4.676 \cdot 10^{-2} \\ 4.195 \cdot 10^{-3} & -4.676 \cdot 10^{-2} & 9.063 \cdot 10^{-2} \end{pmatrix} \cdot \frac{1}{GPa}$$

2.19 $\underline{S}_{11} = S_{22}$ $\underline{S}_{12} = S_{12}$ $\underline{S}_{22} = S_{11}$

$\underline{S}_{16} = 0$ $\underline{S}_{26} = 0$ $\underline{S}_{66} = S_{66}$

The values of c and s are interchanged for 0° and 90° laminas. The values of $\underline{S}_{11}$ and $\underline{S}_{22}$ are interchanged also since the local axes for the 0° lamina are rotated 90°.

2.20

A)

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} 196.4 \\ 126.0 \\ -348.6 \end{pmatrix} \cdot \frac{\mu m}{m}$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix} = \begin{pmatrix} -9.800 \cdot 10^{-2} \\ 6.098 \\ 6.340 \cdot 10^{-1} \end{pmatrix} \cdot MPa$$

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} -7.36 \\ 329.73 \\ 113.40 \end{pmatrix} \cdot \frac{\mu m}{m}$$

c) Principal normal stresses produced by applied global stresses -

$$\sigma_{max} := \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_{max} = 6.162 \cdot MPa$$

$$\sigma_{min} := \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_{min} = -0.1623 \cdot MPa$$

Orientation of maximum principal stress -

$$\theta_{p\sigma} := \frac{1}{2} \cdot atan\left(\frac{2 \cdot \tau_{xy}}{\sigma_x - \sigma_y}\right) \cdot \left(\frac{180 \cdot °}{\pi}\right) \qquad \theta_{p\sigma} = -35.78 \cdot °$$

Principal normal strains -

$$\varepsilon_{max} := \frac{\varepsilon_x + \varepsilon_y}{2} + \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \varepsilon_{max} = 339.0 \cdot \frac{\mu m}{m}$$

$$\varepsilon_{min} := \frac{\varepsilon_x + \varepsilon_y}{2} - \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \varepsilon_{min} = -16.64 \cdot \frac{\mu m}{m}$$

Orientation of maximum principal strain -

$$\theta_{p\varepsilon} := \frac{1}{2} \cdot atan\left(\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}\right) \cdot \left(\frac{180 \cdot °}{\pi}\right) \qquad \theta_{p\varepsilon} = -39.30 \cdot °$$

d) Maximum shear produced by applied global stresses -

$$\tau_{max} := \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \tau_{max} = 3.162 \cdot MPa$$

Orientation of maximum shear -

$$\theta_{s\tau} := \frac{1}{2} \cdot atan\left(-\frac{\sigma_x - \sigma_y}{2 \cdot \tau_{xy}}\right) \cdot \left(\frac{180 \cdot °}{\pi}\right) \qquad \theta_{s\tau} = 9.22 \cdot °$$

Maximum shear strain -

$$\gamma_{max} := 2 \cdot \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \gamma_{max} = 355.7 \cdot \frac{\mu m}{m}$$

Orientation of maximum shear strain -

$$\theta_{s\gamma} := \frac{1}{2} \cdot atan\left(-\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}}\right) \cdot \left(\frac{180 \cdot °}{\pi}\right) \qquad \theta_{s\gamma} = 5.70 \cdot °$$

2.21 $\theta = 34.98 \cdot °$

2.22

$E_x = 2.272 \cdot Msi$

$\nu_{xy} = 0.3632$

$m_x = -0.8530$

$E_y = 3.696 \cdot Msi$

$m_y = 9.538$

$G_{xy} = 1.6 \cdot Msi$

2.23

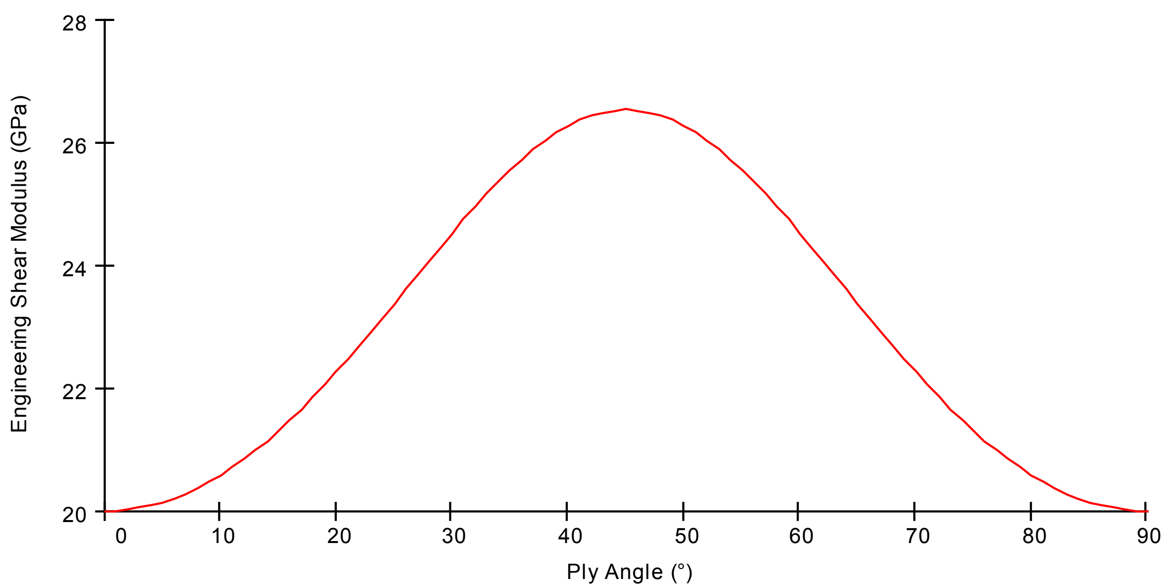
The maximum value of G_{xy} occurs at $\theta = 45°$

$G_{xy}(\theta_3) = 26.54 \cdot GPa$

The minimum value of G_{xy} occurs at $\theta = 0°$

$G_{xy}(\theta_1) = 20.00 \cdot GPa$

Displayed graphically -



2.24 a) Graphite/Epoxy lamina

$\varepsilon_a = 11.60 \cdot \%$

b) Aluminum plate

$$\varepsilon_b = 0.36 \cdot \%$$

2.25 Given

a) Elastic modulus in x direction per Equation (2.112) -

$$G_{12} = 10.10 \cdot GPa$$

b) Elastic modulus -

$$E_x = 13.4 \cdot GPa$$

2.26

$$E_x(30 \cdot °) = 4.173 \cdot Msi$$

b) Only $\frac{1}{G_{12}} - \frac{2 \cdot \nu_{12}}{E_1}$ can be determined, G_{12} and ν_{12} cannot be determined individually.

2.27 Yes, for some values of θ, $E_x < E_2$ if

$$E_2 < E_1 \quad \text{and} \quad G_{12} < \frac{E_1}{2 \cdot \left[\left(\frac{E_1}{E_2} \right) + \nu_{12} \right]}$$

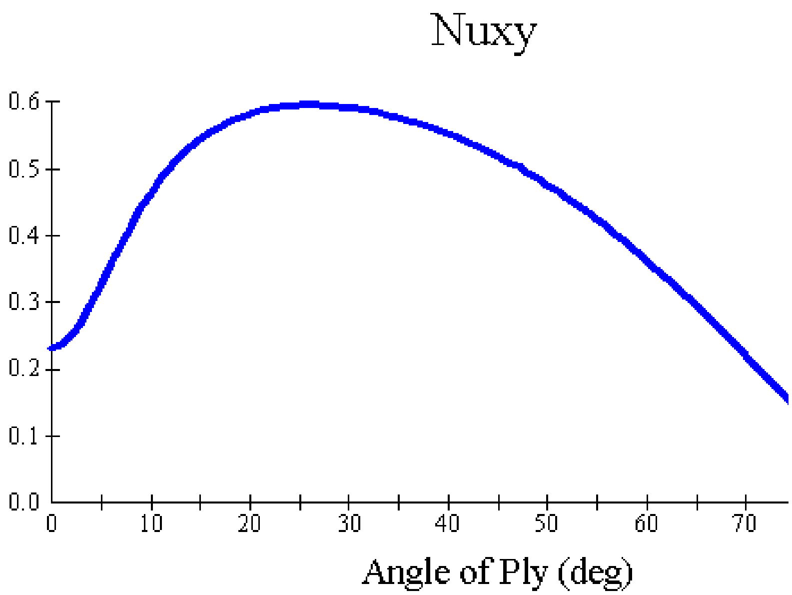
2.28 Yes, for some values of θ, $E_x > E_1$ if

$$G_{12} > \frac{E_1}{2 \cdot (1 + \nu_{12})}$$

2.29 Basing calculations on a Boron/Epoxy lamina

The value of ν_{xy} is maximum for a 26° lamina.

Displayed graphically –

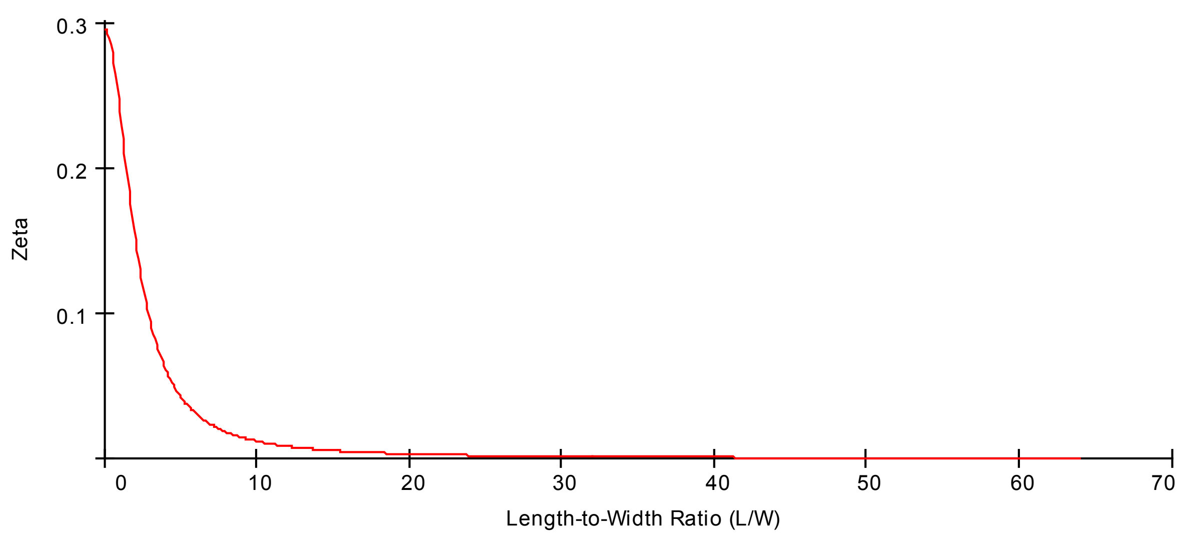


2.30 Given

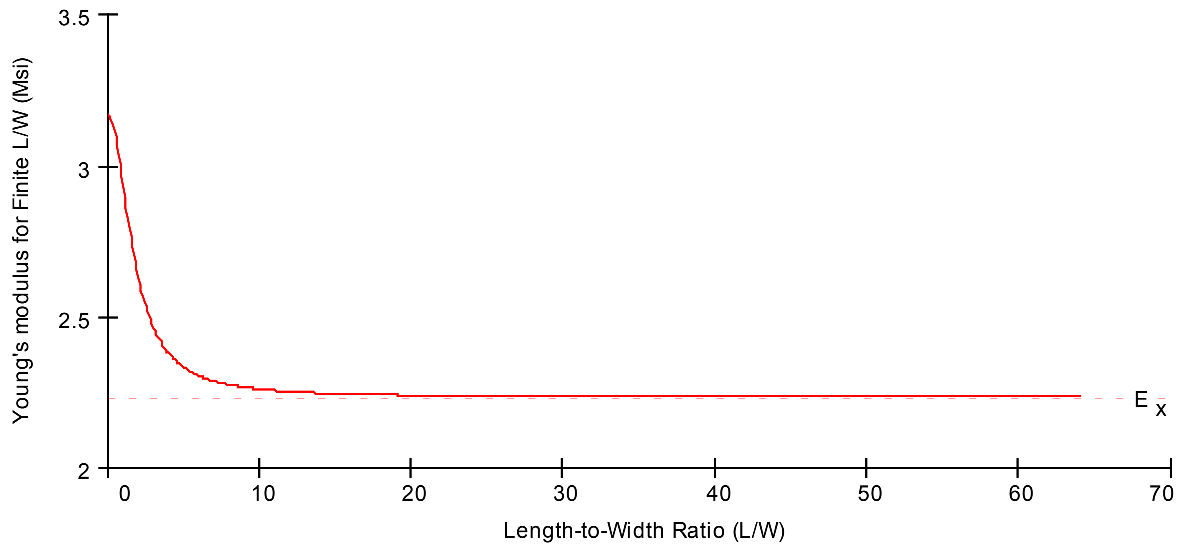
Tabulation:

LWR	ζ	$E1_x$ (Msi)
2	1.504E-1	2.629
8	1.803E-2	2.275
16	4.724E-3	2.245
64	2.998E-4	2.235

Graph of ζ as a function of Length-to-Width ratio -



Graph of Engineering modulus for Finite LWR as a function of ζ -



2.32

$$G_{12} = 0.0488 \cdot \text{GPa}$$

b) From 35° ply data, one can find $\underline{S}_{11}$ and $\underline{S}_{12}$ for the lamina. However, for Equations (2.101a) and (2.101b), there are four unknowns: S_{11}, S_{12}, S_{22}, and S_{66}. Therefore, one cannot find S_{66} nor, in turn, the shear modulus, G_{12}. Although the same is true in the case of the 45° ply, no manipulation of Equations (2.101a) and (2.101b) will allow S_{66} to be expressed in terms of $\underline{S}_{11}$ and $\underline{S}_{12}$ for the 35° ply.

2.33 Elastic properties of Boron/Epoxy from Table 2.2 -

$$U_1 = 12.72 \cdot \text{Msi}$$

$$U_2 = 13.52 \cdot \text{Msi}$$

$$U_3 = 3.493 \cdot \text{Msi}$$

$$U_4 = 4.113 \cdot \text{Msi}$$

$$V_1 = 3.046 \cdot 10^{-1} \cdot \frac{1}{\text{Msi}}$$

$$V_2 = -1.695 \cdot 10^{-1} \cdot \frac{1}{\text{Msi}}$$

$$V_3 = -1.014 \cdot 10^{-1} \cdot \frac{1}{\text{Msi}}$$

$$V_4 = -1.091 \cdot 10^{-1} \cdot \frac{1}{\text{Msi}}$$

2.35 Given

Failure Criterion	**Magnitude of Maximum Positive Shear Stress**	**Magnitude of Maximum Negative Shear Stress**	**Off-axis Shear Strength**
Maximum Stress	134 MPa	70.44 MPa	70.44 MPa
Maximum Strain:	134 MPa	68.99 MPa	68.99 MPa
Tsai-Hill:	62.24 MPa	62.24 MPa	62.24 MPa
Tsai-Wu (Mises-Hencky criterion)	139 MPa	59.66 MPa	59.66 MPa

2.36 The maximum stress failure theory gives the mode of failure. The Tsai-Wu failure theory is a unified theory and gives no indication of the failure mode.

2.37 The Tsai-Wu failure theory agrees closely with experimentally obtained results. The difference between the maximum stress failure theory and experimental results are quite pronounced.

2.38 Given

Maximum Strain: $\sigma_{M\varepsilon} := -249.9 \cdot MPa$

Tsai-Wu (Mises-Hencky criterion): $\sigma_{TWMH} := -259.9 \cdot MPa$

The maximum biaxial stress that can be applied to a 60° lamina of Graphite/Epoxy is conservatively estimated at -249.9 MPa.

2.40 Given the strength parameters for an isotropic material

$$\sigma_1^2 + \sigma_2^2 + 6.25 \cdot \tau_{12}^2 - 1. \cdot \sigma_1 \cdot \sigma_2 < \sigma_T^2$$

2.41 Given the strength parameters for a unidirectional Boron/Epoxy system -

Since $H_{12}^2 < H_{11} \cdot H_{22}$ the stability criterion is satisfied.

2.42 The units for the coefficient of thermal expansion in the USCS system are in/in/°F. In the SI system the units for the coefficient of thermal expansion are m/m/°C.

2.43

$$\begin{pmatrix} \varepsilon_{C1} \\ \varepsilon_{C2} \\ \gamma_{C12} \end{pmatrix} = \begin{pmatrix} 0 \\ 1200 \\ 0 \end{pmatrix} \cdot \frac{\mu m}{m}$$

$$\Delta T_{2\_Offset} = -54.30 \cdot °C$$

2.44

$$\begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{pmatrix} = \begin{pmatrix} 10.403 \\ 6.653 \\ -6.495 \end{pmatrix} \cdot \frac{\frac{\mu in}{in}}{°F}$$

2.45 The units for the coefficient of moisture expansion in the USCS system are in/in/lbm/lbm. In the SI system the units for the coefficient of moisture expansion are m/m/kg/kg.

2.46

$$\begin{pmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{pmatrix} = \begin{pmatrix} 0.4500 \\ 0.1500 \\ -0.5196 \end{pmatrix} \cdot \frac{\frac{m}{m}}{\frac{kg}{kg}}$$