INTERSECTION OF SPIRAL CURVE WITH CIRCLE[a]

Discussion by Thomas G. Davis[5] and Sum Lin[6]

The discussers suggest that intersections on cornu spirals or clothoids may be found more easily by solving for the parameter l and backcomputing to obtain Cartesian coordinates.

Expanding (1*c*) into Maclaurin series and integrating term-by-term [cf. (2)]

$$x(l) = l\sum_{n=0}^{\infty}\frac{(-1)^n\tau^{2n}}{(4n+1)(2n)!};\quad y(l) = l\sum_{n=0}^{\infty}\frac{(-1)^n\tau^{2n+1}}{(4n+3)(2n+1)!} \qquad (24)$$

where, as in (1*b*), $\tau = (l/A)^2/2$; and x and y = Cartesian coordinates at spiral length l. These expressions are suitable for computer coding; the number of terms used determines the accuracy of results.

Eq. (5) may be written as

$$[x(l) - a]^2 + [y(l) - b]^2 = R^2 \qquad (25)$$

or [cf. (6)]

$$G(l) = [x(l) - a]^2 + [y(l) - b]^2 - R^2 = 0 \qquad (26)$$

The roots of G, i.e. particular values of l such that (26) is satisfied, are the required intersections. Cartesian coordinates of these intersections are given by evaluating (24) at the roots.

The derivative of G is

$$G'(l) = 2\{x'(l)[x(l) - a] + y'(l)[y(l) - b]\} \qquad (27)$$

Now, $x'(l)$ and $y'(l)$ are derivatives of the integrals (1*c*), i.e. they are the integrands

$$x'(l) = \cos\tau;\quad y'(l) = \sin\tau \qquad (28)$$

It is neither necessary nor desirable to represent these derivatives as series. Substituting (28) into (27)

$$G'(l) = 2\{[x(l) - a]\cos\tau + [y(l) - b]\sin\tau\} \qquad (29)$$

Unlike (8), (29) is continuous for all values of l with any choice of constants a, b, and A. The roots of (26) may be found by the iteration

$$l_{i+1} = l_i - \frac{G(l_i)}{G'(l_i)} \qquad (30)$$

There are no exceptional cases to consider.

These parametric equations, and others like them, have long been used in the Civil Engineering Automation Library (CEAL) (Miller 1995) to solve for intersections of offsets to spirals with lines, circles, and other offsets to spirals, as well as orthogonal and tangent point projections onto spirals. In every

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[a] February 1995, Vol. 121, No. 1, by Olcay Öztan, Orhan Baykal, Oguz Müftüoglu, and Muhammed Sahin (Paper 7081).

[5] Appl. Mathematician, CLM/Systems, Inc., 5601 Mariner Dr., Tampa, FL 33609.

[6] Sr. Applications Engr., Geopak Corp., 1190 N.E. 163rd St., N. Miami Beach, FL 33162.

instance, solutions are calculated using Newton's method to determine the spiral length l. The case of spiral-intersect-spiral is solved using Newton's method for nonliner systems; the other cases require the solution of a single, nonlinear equation. Results are obtained in less than 1 s on a typical Intel 80486 microcomputer and show excellent agreement with numerical solutions generated by Maple V (Char et al. 1991) using intrinsic Fresnel integral functions.

The authors have considered the cases of two, one, or no intersections of the spiral and circle. There remains the case of three intersections. Since any three noncollinear points determine a unique circle, and three distinct spiral points ($0 \leq \tau \leq 3\pi/2$) are never collinear, three intersections are always possible, however short the spiral length. If distinct intersections are found by using upper and lower bounds as initial values, a search should be made between these solutions for yet another root.

If the circle is tangent to the spiral at a particular spiral length l^* with corresponding spiral angle $\tau(l^*) = \tau^*$, then

$$a = x(l^*) - R \sin \tau^*; \quad b = y(l^*) + R \cos \tau^* \tag{31}$$

where R is positive when measured from the spiral toward its evolute, and negative otherwise. The value of R is arbitrary; whether or not the circle is an osculating circle of the spiral is immaterial. Substituting (31) into (26) and (29)

$$G(l^*) = R^2 \sin^2\tau^* + R^2 \cos^2\tau^* - R^2 \equiv 0 \tag{32}$$

and

$$G'(l^*) = 2(R \sin \tau^* \cos \tau^* - R \cos \tau^* \sin \tau^*) \equiv 0 \tag{33}$$

Eqs. (6) and (8) are also identically zero at the point of tangency. This potentially troublesome condition is encountered whenever multiple roots are sought using Newton's method. While the authors have presented an interesting perturbation technique to induce convergence in (7), (30) has been found to converge without difficulty.

The limit $A = l = r$ for a spiral route element may not be adequate to encompass the "connecting" spiral, or oval curve (Baass 1984), that joins two circles. Eqs. (24), (26), (29), and (30) are applicable to any portion of the clothoid k_1 shown in Fig. 1.

Finally, it should be noted that the authors have apparently neglected to account for negating the ± operator throughout the paper. A $-(\pm)$ or $\mp$ operator should appear in (6), the numerator of (9), and (20). This is a mathematical necessity rather than a matter of convention.

It is the opinion of the discussers that on page 3, (1*b*) should read

$$\tau = \frac{1}{2}\left(\frac{A}{r}\right)^2 = \frac{1}{2}\left(\frac{l}{A}\right)^2$$

On page 5, (6) should read

$$F(x_s) = \frac{x_s^3}{6A^2}\left[1 - 0.205\left(\frac{x_s}{A}\right)^4\right]^{-0.27875} \mp \sqrt{R^2 - (x_s - a)^2} - b = 0$$

On page 6, (9) should read

$$x_{i+1} = x_i - \left\{\frac{x_i^3}{6A^2}\left[1 - 0.205\left(\frac{x_i}{A}\right)^4\right]^{-0.27875} \mp \sqrt{R^2 - (x_i - a)^2} - b\right\} \Big/ \left\{\frac{1}{2}\left(\frac{x_i}{A}\right)^2\left[1 - 0.27371\left(\frac{x_i}{A}\right)^4\right]^{-0.487134} \pm \frac{(x_i - a)}{\sqrt{R^2 - (x_i - a)^2}}\right\}$$

On page 6, (10) should read

$$-\left(\frac{dY}{dX}\right)_{X=a\pm R} = \pm\left[\frac{(x_i - a)}{\sqrt{R^2 - (x_i - a)^2}}\right]_{x_i a\pm R} = \pm\infty$$

On page 8, (20) should read

$$Y'(X_i) = \mp\frac{(X_i - a)}{\sqrt{R^2 - (X_i - a)^2}}$$

APPENDIX. REFERENCES

Baass, K. G. (1984). "The use of clothoid templates in highway design." *Transp. Forum*, 1(3), 47–52.

Char, B. W., Geddes, K. O., Gonnet, G. H., Leong, B. L., Monagan, M. B., and Watt, S. M. (1991). *Maple V language reference manual.* Springer-Verlag, New York, N.Y.

Miller, C. L. (1995). *CEAL user's manual; release 7.0.* CLM/Systems, Tampa, Fla.

Discussion by K. S. Li[7]

The authors have presented an iterative procedure for calculating the intersection point of a spiral curve with a circle. The proposed method is based on Newton-Raphson's method for solving the root(s) of an equation of the form

$$F(t) = 0 \tag{34}$$

where t is the dependent variable. If (34) can be formulated in an alternative form

$$t = f(t) \tag{35}$$

the root(s) of (34) can be solved by means of the following iterative scheme:

$$t_i = f(t_{i-1}) \tag{36}$$

where t_i = ith estimate of the root of (34). The simple iterative scheme can be started by providing an initial estimate of $t = t_0$.

The convergence rate of the simple iterative scheme of (36) can be greatly enhanced using the following compound iterative scheme developed by Li and White (1987):

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[7] Dir., Mitchell McFarlane Brentnall & Partners, 17/F, National Mutual Centre, Wanchai, Hong Kong.