CONFIDENCE REGION RADIUS

Univariate Normal Distribution

For a normally distributed variable with zero mean, the probability that an observation will be contained in the interval [-R,R] is

$$P = \frac{2}{\sigma\sqrt{2\pi}} \int_0^R \exp\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right) dx\,. \tag{1}$$

Evaluating (1)

$$P = \operatorname{erf}\left(\frac{1}{\sqrt{2}}\frac{R}{\sigma}\right) \Rightarrow R = \sqrt{2}\,\sigma\,\operatorname{erf}^{-1}(P) \tag{2}$$

where erf is the error function.

Bivariate Normal Distribution

For uncorrelated, normally distributed variables with zero mean, the probability that an observation will be contained in area A is

$$P = \frac{1}{\sigma_x\sigma_y 2\pi} \iint_A \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\right] dA\,. \tag{3}$$

Casting (3) in polar coordinates and integrating over the disk centered at the origin with radius R

$$P = \frac{4}{\sigma_x\sigma_y 2\pi} \int_0^R \int_0^{\frac{\pi}{2}} r\,\exp\left[-\frac{r^2}{2}\left(\frac{\cos^2\theta}{\sigma_x^2} + \frac{\sin^2\theta}{\sigma_y^2}\right)\right]\, d\theta\, dr\,. \tag{4}$$

Substituting $\sin^2\theta = 1 - \cos^2\theta$

$$P = \frac{2}{\sigma_x\sigma_y\pi} \int_0^R r\,\exp\left(-\frac{1}{2}\frac{r^2}{\sigma_y^2}\right) \int_0^{\frac{\pi}{2}} \exp(-2\,a\,r^2\cos^2\theta)\, d\theta\, dr \tag{5}$$

where $a = (1/\sigma_x^2 - 1/\sigma_y^2)/4$.

Evaluating the inner integral of (5)

$$P = \frac{1}{\sigma_x \sigma_y} \int_0^R r \exp(b\, r^2)\ \ I_0(a\, r^2)\, dr \tag{6}$$

where $b = -(1/\sigma_x^2 + 1/\sigma_y^2)/4$, and I_0 is the zero order modified Bessel function of the first kind.

When $\sigma_x = \sigma_y = \sigma$

$$P = 1 - \exp\left(-\frac{1}{2}\frac{R^2}{\sigma^2}\right) \Rightarrow R = \sigma\sqrt{-2\ln(1-P)}\,. \tag{7}$$

Trivariate Normal Distribution

For uncorrelated, normally distributed variables with zero mean, the probability that an observation will be contained in volume V is

$$P = \frac{1}{\sigma_x \sigma_y \sigma_z (2\pi)^{3/2}} \iiint_V \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2}\right)\right] dV\,. \tag{8}$$

Casting (8) in cylindrical coordinates and integrating over the ball centered at the origin with radius R

$$P = \frac{8 \int_0^R \int_0^{\frac{\pi}{2}} r \exp\left[-\frac{r^2}{2}\left(\frac{\cos^2\theta}{\sigma_x^2} + \frac{\sin^2\theta}{\sigma_y^2}\right)\right] \int_0^{\sqrt{R^2-r^2}} \exp\left(-\frac{1}{2}\frac{z^2}{\sigma_z^2}\right) dz\, d\theta\, dr}{\sigma_x \sigma_y \sigma_z (2\pi)^{3/2}}\,. \tag{9}$$

Evaluating the innermost integral of (9)

$$P = \frac{2}{\sigma_x \sigma_y \pi} \int_0^R \int_0^{\frac{\pi}{2}} r \exp\left[-\frac{r^2}{2}\left(\frac{\cos^2\theta}{\sigma_x^2} + \frac{\sin^2\theta}{\sigma_y^2}\right)\right] \operatorname{erf}\left(\frac{\sqrt{R^2 - r^2}}{\sqrt{2}\,\sigma_z}\right) d\theta\, dr\,. \tag{10}$$

Substituting $\sin^2\theta = 1 - \cos^2\theta$

$$P = \frac{2}{\sigma_x \sigma_y \pi} \int_0^R r \operatorname{erf}\left(\frac{\sqrt{R^2 - r^2}}{\sqrt{2}\,\sigma_z}\right) \exp\left(-\frac{1}{2}\frac{r^2}{\sigma_y^2}\right) \int_0^{\frac{\pi}{2}} \exp(-2\, a\, r^2 \cos^2\theta)\, d\theta\, dr \tag{11}$$

where $a = (1/\sigma_x^2 - 1/\sigma_y^2)/4$.

Evaluating the inner integral of (11)

$$P = \frac{1}{\sigma_x \sigma_y} \int_0^R r \, \mathrm{erf}\left(\frac{\sqrt{R^2 - r^2}}{\sqrt{2}\, \sigma_z} \right) \exp(b\, r^2) \;\; I_0(a\, r^2)\, dr \qquad (12)$$

where $b = -(1/\sigma_x^2 + 1/\sigma_y^2)/4$.

When $\sigma_x = \sigma_y = \sigma_h$

$$P = \frac{1}{\sigma_h^2} \int_0^R r \, \mathrm{erf}\left(\frac{\sqrt{R^2 - r^2}}{\sqrt{2}\, \sigma_z} \right) \exp\left(-\frac{1}{2} \frac{r^2}{\sigma_h^2} \right) dr\,. \qquad (13)$$

When $\sigma_x = \sigma_y = \sigma_h$ and $\sigma_h > \sigma_z$

$$P = \mathrm{erf}\left(\frac{1}{\sqrt{2}} \frac{R}{\sigma_z} \right) - \frac{\sigma_h}{\sqrt{c}} \exp\left(-\frac{1}{2} \frac{R^2}{\sigma_h^2} \right) \mathrm{erf}\left(\sqrt{\frac{c}{2}} \frac{R}{\sigma_h \sigma_z} \right) \qquad (14)$$

where $c = \sigma_h^2 - \sigma_z^2 > 0$.

When $\sigma_x = \sigma_y = \sigma_z = \sigma$

$$P = \mathrm{erf}\left(\frac{1}{\sqrt{2}} \frac{R}{\sigma} \right) - \sqrt{\frac{2}{\pi}} \frac{R}{\sigma} \exp\left(-\frac{1}{2} \frac{R^2}{\sigma^2} \right). \qquad (15)$$

Davis and Kleder (2006) solve (6) and (12) for radius R corresponding to probability P. The algorithm is an extension of code presented by Kleder (2004) with enhancements suggested by Koopman (2004).

References

Davis, T. and Kleder, M. (2006). “Confidence region radius.” (http://www.eng.usf.edu/~tdavis/surveying/crr.m)

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